# Math 225A – Model Theory

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## General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

## Lecture 5

We start with some examples of first-order theories.

*Example* (Peano Arithmetic). Peano arithmetic is intended as a formalization of the laws of arithmetic on the natural numbers. Let $\tau = \{0, 1, +, \cdot, <\}$ where 0 and 1 are constants, + and $\cdot$ are binary function symbols, and $<$ is a binary relation. The axioms of this theory can be presented in many different ways. There are basically two sorts of axioms: the ones describing the algebraic properties of the natural numbers and the ones describing induction. The algebraic axioms basically state (in the formal language $\mathscr{L}(\tau)$) that $(\mathbb{N}, 0, 1, +, \cdot, <)$ is a discretely ordered semi-ring. They may be stated as follows.

- $\forall x\ x + 0 = x$
- $\forall x \forall y\ (x + y) + 1 = x + (y + 1)$
- $\forall x \forall y\ [(x + 1 = y + 1) \to x = y]$
- $\forall x\ \neg(x + 1 = 0)$
- $\forall x\ x \cdot 0 = 0$
- $\forall x \forall y\ x \cdot (y + 1) = x \cdot y + x$
- $\forall x\ \neg(x < x)$
- $\forall x \forall y\ (x < y) \vee (x = y) \vee (y < x)$
- $\forall x \forall y \forall z\ (x < y \wedge y < z \to x < y)$
- $\forall x \forall y \forall z\ x < y \to x + z < y + z$
- $\forall x \forall y\ (x < y + 1) \to (x < y \vee x = y)$

This takes care of the algebraic axioms. Note that the above list is a finite list of sentences, and so we could take the conjunction over all of them and write them as a single sentence in $\mathscr{L}(\tau)$. This is not the case for the induction axioms. Induction is given by a *schema* of axioms. For each formula $\varphi(x, \bar{y})$ in $\mathscr{L}(\tau)$ we have the axiom, $I(\varphi)$:

$$\forall \bar{y}\ \ ([\varphi(0, \bar{y}) \wedge \forall z\ (\varphi(z, \bar{y}) \to \varphi(z + 1, \bar{y}))] \to \forall x\ \varphi(x, \bar{y}))$$

i.e. if $\varphi(x, \bar{y})$ is true of 0 and if each time $\varphi(z, \bar{y})$ is true then so is $\varphi(z+1, \bar{y})$, then $\varphi(x, \bar{y})$ is true for all $x$.

The collection of all these infinitely many axioms (both the algebraic and the inductive) is called the theory of **Peano arithmetic**, or $PA$. It is of course meant to axiomatize $\mathbb{N}$. There are however many other *non-standard* models of Peano arithmetic. A theorem of Tenenbaum however says that no other models of Peano arithmetic has a recursive presentation. In a sense this means that one will never "see" any of the non-standard models of $PA$.

*Example* (Orders). Let $\tau = \{<\}$ be the signature of a single binary relation symbol. The **partial orders** are $\tau$-structures satisfying the axioms

- $\forall x \, \neg(x < x)$
- $\forall x \forall y \forall z \, (x < y \wedge y < z) \to (x < z)$

The theory of **linear orders** is a sub theory of the theory of partial orders, i.e. it contains the two axioms above and the extra axiom

$$\forall x \forall y (x < y \vee x = y \vee y < x).$$

The theory of **well-orders** is the theory of linear orders together with the statement that

for every nonempty subset $X$ there exists a least element $a \in X$.

This last statement is not a first-order statement since we are quantifying over both subsets and elements of the subsets. So this is a second-order statement and can be made rigorous in second-order logic.

*Exercise.* Show that the class of well-orders is axiomatizable in $\mathscr{L}_{\omega_1\omega_1}$. [Hint: $(A, <)$ is well-ordered iff there are no strict descending chains.]

*Example* (ZFC). **Zermelo-Fraenkel set theory with choice** ($ZFC$) is an axiomatization system for doing set theory. As with $PA$, $ZFC$ is usually given by an axiom schema. In fact $ZFC$ is not finitely axiomatizable.

*Example* (ACF). **Algebraically Closed Fields** ($ACF$). This is again given by a schema of axioms which express that every monic polynomial, of degree $m$ (for each $m \in \omega$) with coefficients in the field, has a solution. $ACF$ is in fact not finitely axiomatizable, but there is an open question which asks whether $ACF$ is finite-variable axiomatizable. I.e. if we allow only finitely many variables in the construction of the language, can we axiomatize $ACF$?

## Preservation of formulas

Fix a signature $\tau$. We work in $\mathscr{L}_{\omega\omega}$ although some of the following makes sense in higher-order logics. We start with the $\forall_n$ (read: "A $n$") and $\exists_n$ (read: "E $n$") hierarchy of formulas. The subscripts refer to the number of alterations of quantifiers there are in a given formula.

**Definition.** The class of $\forall_0$ formulae is the same as the class of $\exists_0$ formulae and they are the quantifier-free formulae. A formula is $\forall_{n+1}$ if it has the form

$$\forall \bar{y} \left( \bigvee \bigwedge \Phi \right)$$

where every $\varphi \in \Phi$ is an $\exists_n$-formula.
A formula is $\exists_{n+1}$-formula if it has the form

$$\exists \bar{y} \left( \bigvee \bigwedge \Phi \right)$$

where each $\varphi \in \Phi$ is a $\forall_n$-formula.

Note that $\bigcup_n \forall_n = \bigcup_n \exists_n$.

*Remark.* Often one says that a formula which is "equivalent" to a formula in a given class is in that class. I.e. if $\varphi$ is "equivalent" to a formula $\psi$ which is $\forall_n$ then we may say that $\varphi$ is $\forall_n$ as well. Here "equivalent" means equivalence modulo some implicit background theory $T$. I.e. $\varphi$ and $\psi$ are **equivalent modulo** $T$ if for all $\mathfrak{A}$ such that $\mathfrak{A} \models T$ then $\mathfrak{A} \models \varphi \leftrightarrow \psi$.

*Remark.* The classification given above is similar to the arithmetical hierarchy in recursion theory (see lecture 4), where statements are divided into $\Sigma^0_n$ and $\Pi^0_n$ classes. However, these hierarchies are different, namely in the lowest level $\Sigma^0_0 = \Pi^0_0$ allows the use of *bounded quantifiers*. Bounded quantifiers are *not* allowed in the $\forall_0$ and $\exists_0$ formulae.

A formula is **prenex** if it consists of a (possibly empty) string of quantifiers followed by a quantifier-free formula.

**Proposition.** *For every $\psi \in \mathscr{L}(\tau)$ there exists $\theta \in \bigcup_n \forall_n = \bigcup \exists_n$ such that $\psi \leftrightarrow \theta$. In words; every formula is equivalent to a formula in prenex normal form.*

*Proof.* We work by induction on the complexity of $\psi$.
If $\psi$ is atomic then it is $\forall_0$ (and $\exists_0$) already. If $\psi = \neg\theta$ then by induction $\theta$ is in, say, $\exists_n$ so $\theta$ has the form $\exists \bar{y} \tilde{\theta}$ with $\tilde{\theta}$ in $\forall_{n-1}$, so $\neg\theta$ is equivalent to $\forall \bar{y} \neg \tilde{\theta}$.
If $\psi = (\varphi \vee \theta)$ then we may assume $\varphi \in \forall_n$ and $\theta \in \forall_n$ then by definition $\psi$ is in $\forall_n \subseteq \exists_{n+1}$ (since we can always put irrelevant quantifiers in from of a formula).
Likewise for $\psi = \exists\theta$, we may assume $\theta \in \forall_n$ then $\psi \in \exists_{n+1}$. By induction we are done. □

We shall see later that this hierarchy is in fact proper, in the sense that the inclusions $\forall_n \subseteq \exists_n$ and $\exists_n \subseteq \forall_{n+1}$ are proper.

*Remark.* This is related to Hilbert's 10th problem; Find an algorithm to decide for $p(x) \in \mathbb{Z}[x]$ whether there exists $\bar{a} \in \mathbb{Z}^n$ such that $p(\bar{a}) = 0$. The Matiyasevich-Davis-Putnam-Robinson (MDPR) theorem states that no such algorithm exists. This problem can then be asked for polynomials over the rational. This is an open problem. But, a recent theorem of Jochen Koenigsmann states that there is a *universal definition* of the integers inside the rationals. I.e. a $\forall_1$-formula $\theta(x)$ such that $\mathbb{Q} \models \theta(a)$ if and only if $a \in \mathbb{Z}$. If there were an $\exists_1$-definition then the MDPR-theorem would imply that there is no algorithm to decide over $\mathbb{Q}$ either.
Later in the course we will prove the following fact: There exist two models $\mathfrak{A}$ and $\mathfrak{B}$ of the theory of the rational, $(\mathbb{Q}, +, \cdot, 0, 1)$, such that $\mathfrak{A} \subseteq \mathfrak{B}$ and there is some polynomial over $\mathfrak{A}$ which has no solutions over $\mathfrak{A}$ but does have solutions over $\mathfrak{B}$.

We shall now look at what kinds of formulae are preserved by certain types of maps.

**Proposition** (Going-up)**.** *If $\iota : \mathfrak{A} \to \mathfrak{B}$ is an embedding and $\varphi(\bar{x})$ is $\exists_1$, then $\mathfrak{A} \models \varphi(\bar{a})$ implies that $\mathfrak{B} \models \varphi(\iota\bar{a})$. Equivalently if $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{B} \models (\iota\bar{a})$.*

*Notation.* If $\varphi(\bar{x})$ is a formula and $\bar{a}$ is a tuple of the same length as $\bar{x}$ then we write $\varphi(\bar{a}/\bar{x})$ for the formula where we have substituted $\bar{a}$ for $\bar{x}$.

*Proof.* Write $\varphi$ as $\exists\bar{y}(\bigvee\bigwedge\Psi)$ where $\Psi$ is a set of $\forall_0$ ( i.e. quantifier-free) formulae. Then if $\mathfrak{A}_{\bar{a}} \models \varphi(\bar{a})$ then there exist $\bar{b}$ from $\mathfrak{A}$ such that $\mathfrak{A}_{\bar{a},\bar{b}} \models \bigvee\bigwedge\Psi(\bar{a}/\bar{x}, \bar{b}/\bar{y})$. We have already shown that if $\theta(\bar{z})$ is quantifier-free and $\mathfrak{A} \subseteq \mathfrak{B}$ then for all $\bar{c}$ in $\mathfrak{A}$ we have $\mathfrak{A} \models \theta(\bar{c})$ if and only if $\mathfrak{B} \models \theta(\bar{c})$. So $\mathfrak{B} \models \bigvee\bigwedge\Psi(\bar{a}/\bar{x}, \bar{b}/\bar{y})$, i.e. $\mathfrak{B} \models \exists\bar{y}\bigvee\bigwedge\Psi(\bar{a}/\bar{x}, \bar{y})$. □

If we weaken the hypothesis and assume only that there is a homomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ we can still get a result. We call a formula $\exists_1^+$ if no negations are involved, i.e. if it has the form $\exists\bar{y}(\bigvee\bigwedge\Phi)$ where all elements of $\Phi$ are atomic.

**Proposition.** *If $\rho : \mathfrak{A} \to \mathfrak{B}$ is a $\tau$-homomorphism and $\varphi$ is $\exists_1^+$, then $\mathfrak{A}_{\bar{a}} \models \varphi(\bar{a})$ implies $\mathfrak{B}_{\rho\bar{a}} \models \varphi(\rho\bar{a})$.*

*Proof.* Immediate from the definition of homomorphism. □

The "Going-up" proposition has a dual "Going-down" proposition.

**Proposition** (Going-down)**.** *If $\mathfrak{A} \subseteq \mathfrak{B}$ and $\varphi$ is $\forall_1$ then $\mathfrak{B} \models \varphi(\bar{a})$ implies that $\mathfrak{A} \models \varphi(\bar{a})$.*

*Proof.* We note that $\varphi$ is equivalent to a formula of form $\neg\exists\neg$. Then apply the "going-up" proposition. □

As a nice consequence of this propositions, suppose $T$ is a theory where all axioms of $T$ are $\forall_1$. Then the class of models is closed under formation of substructures. I.e. if $\mathfrak{A} \models T$ then all substructures of $\mathfrak{A}$ also model $T$. Dually, if $T$ is a theory all of whose sentences are $\exists_1$ then, by the going-up proposition, the class of models of $T$ is closed under formation of superstructures. I.e. if $\mathfrak{A} \models T$ and $\mathfrak{B}$ is some superstructure of $\mathfrak{A}$ then $\mathfrak{B} \models T$ as well.
We shall in fact see that these characterizations of universal and existential theories have converses. That is, if a theory $T$ has the property that whenever $\mathfrak{A} \models T$ then for all substructures $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B} \models T$, then $T$ is universal. Similarly if $T$ has the property that whenever $\mathfrak{A} \models T$ and $\mathfrak{B} \supseteq \mathfrak{A}$ then $\mathfrak{B} \models T$, then $T$ is existential.

We now turn to situations where we can preserve $\forall_2$-sentences.

**Definition.** A **chain** of models is a sequence $(\mathfrak{A}_i)_{i\in I}$ of $\tau$-structures such that $(I, <)$ is totally ordered and such that

$$i < j \Rightarrow \mathfrak{A}_i \subseteq \mathfrak{A}_j.$$

Given a chain $(\mathfrak{A}_i)_{i\in I}$ of $\tau$-structures we can form the **direct limit**[1]

$$\tilde{\mathfrak{A}} = \bigcup_{i\in I} \mathfrak{A}_i.$$

The domain of $\tilde{\mathfrak{A}}$ will be the union $\bigcup_{i\in I} \mathrm{dom}(\mathfrak{A}_i)$. The interpretations of the symbols will be as follows.

- for $c \in \mathcal{C}_\tau$ let $c^{\tilde{\mathfrak{A}}} = c^{\mathfrak{A}_i}$ for any choice of $\mathfrak{A}_i$
- for $f \in \mathcal{C}_\tau$ then $f^{\tilde{\mathfrak{A}}} = \bigcup_{i\in I} f^{\mathfrak{A}_i}$
- for $R \in \mathcal{R}_\tau$ then $R^{\tilde{\mathfrak{A}}} = \bigcup_{i\in I} f^{\mathfrak{A}_i}$.

All these choices are well-defined and since $\mathfrak{A}_i \subseteq \mathfrak{A}_j$ whenever $i < j$ we get that $\mathfrak{A}_i \subseteq \tilde{\mathfrak{A}}$ for all $i \in I$.

$\forall_2$ sentences "go up" in chains.

**Proposition.** *If $\varphi$ is $\forall_2$ and if for all $i \in I$ $\mathfrak{A}_i \models \varphi$ then $\tilde{\mathfrak{A}} \models \varphi$.*

*Proof.* We can write $\varphi$ as $\forall\bar{x}\exists\bar{y}\ \theta$ with $\theta$ quantifier-free. Let $\bar{a}$ be a sequence from $\tilde{\mathfrak{A}}$. Since $\bar{a}$ is finite there exists $i \in I$ such that $\bar{a}$ comes from $\mathrm{dom}(\mathfrak{A}_i)$. Now since $\mathfrak{A}_i \models \varphi$ we have

$$\mathfrak{A}_{i,\bar{a}} \models \exists\bar{y}\ \theta(\bar{a}/x)$$

[1] It is in fact a direct limit in the category theoretic sense.

and so by the going-up for $\exists_1$,

$$\tilde{\mathfrak{A}}_{\bar{a}} \models \exists \bar{y}\ \theta(\bar{a}/x)$$

and since this was true for any choice of $\bar{a}$ it follows that

$$\tilde{\mathfrak{A}} \models \varphi.$$

□

The converse is also true, i.e. a theory $T$ admits an $\forall_2$-axiomatization if and only if it is preserved under unions of chains.