Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 13

Last time we defined a topology on $\mathrm{Sym}(X)$. We showed that the automorphism group of a structure, $\mathrm{Aut}(\mathfrak{A})$, is a closed subgroup of the permutation group of the domain, $\mathrm{Sym}(A)$. It follows that if $\mathfrak{A}^+$ is an expansion of $\mathfrak{A}$ then $\mathrm{Aut}(\mathfrak{A}^+)$ is a closed subgroup of $\mathrm{Aut}(\mathfrak{A})$ with respect to the subspace topology on $\mathrm{Aut}(\mathfrak{A})$. This has a converse. Any closed subgroup of $\mathrm{Aut}(\mathfrak{A})$ can be realized as the automorphism group of an extension of $\mathfrak{A}$.

Proposition. *Let $\mathfrak{A}$ be a τ-structre and H a closed subgroup of* $\mathrm{Aut}(\mathfrak{A})$*. Then there is an extension of signatures $\tau^+ \supseteq \tau$ and an extension $\mathfrak{A}^+$ of $\mathfrak{A}$ to τ^+, such that $H = \mathrm{Aut}(\mathfrak{A}^+)$.*

Proof. The action of H on the set $A = \mathrm{dom}(\mathfrak{A})$ is used to determine some new relation symbols. For each $n \in \omega$ and for each H-orbit $X \subseteq A^n$ let $R_X \in \mathcal{R}_{\tau^+}$ be a new relation symbol of arity n. We let $\mathfrak{A}^+$ be the extension of $\mathfrak{A}$ where

$$R_X^{\mathfrak{A}^+} := X.$$

We claim $H = \mathrm{Aut}(\mathfrak{A}^+)$.

First let $h \in H$. Let $R_X \in \tau^+$ be one of the new relation symbols, and let $a \in A^n$ such that $\mathfrak{A}^+ \models R_X(a)$. Then $X = H.a \subseteq A^n$. So $h.a \in X$, i.e. $\mathfrak{A}^+ \models R_X(ha)$. Conversely if $\mathfrak{A} \models R_X(ha)$ then $ha \in X$ and so $a = h^{-1}(ha) \in X$ as well. Since $h \in \mathrm{Aut}(\mathfrak{A})$ and h fixes all new relation symbols we have that $h \in \mathrm{Aut}(\mathfrak{A}^+)$. So $H \leq \mathrm{Aut}(\mathfrak{A}^+)$.

Now let $\sigma \in \mathrm{Aut}(\mathfrak{A}^+)$. We will show that σ is in $\overline{H}$ and so by assumption in H (since H is closed). Let $U \ni \sigma$ be an open set. We may assume $U = U_{a,b}$ for some $a, b \in A^n$ so that $\sigma a = b$. Let $X = Ha$ be the H-orbit of a. Since $\sigma \in \mathrm{Aut}(\mathfrak{A}^+)$ we have

$$\mathfrak{A}^+ \models R_X(a) \Longrightarrow \mathfrak{A}^+ \models R_X(\sigma a)$$

so $\mathfrak{A}^+ \models R_X(b)$ i.e. $b \in X$. So there is some $h \in H$ such that $ha = b$. But this means that $h \in H \cap U_{a,b}$. In particular $H \cap U_{a,b} \neq \emptyset$, so every open set containing σ meets H, so $\sigma \in \overline{H} = H$. □

Remark. As previously mentioned, for a τ-structure $\mathfrak{A}$ and $a \in A^n$ the type of a written $\mathrm{tp}(a)$ is defined to be

$$\mathrm{tp}(a) := \mathrm{Th}(\mathfrak{A}_a) = \{\varphi(\bar{x}) \in \mathscr{L}(\tau) : \mathfrak{A} \models \varphi(\bar{a})\}$$

i.e. all formulae which are true (in $\mathfrak{A}$) of the tuple a.

If there is some $\sigma \in \mathrm{Aut}(\mathfrak{A})$ and $a, b \in A^n$ with $\sigma a = b$ then $\mathrm{tp}(a) = \mathrm{tp}(b)$.

The converse is true in the structure $\mathfrak{A}^+$ constructed in the above proof. I.e. $\mathrm{tp}(a) = \mathrm{tp}(b)$ if and only if there is some $\sigma \in \mathrm{Aut}(\mathfrak{A}^+)$ such that $\sigma a = b$. For general τ-structures, $\mathfrak{B}$, this is not the case. There may be tuples a and b with $\mathrm{tp}(a) = \mathrm{tp}(b)$ and yet $\mathrm{Aut}(\mathfrak{B})a \neq \mathrm{Aut}(\mathfrak{B})b$.

Example. Let $\tau = \{E\}$ be the theory of a single equivalence relation. Let $\mathfrak{A}$ be a τ-structure such that there are exactly two equivalence classes, one of size $\aleph_0$ and the other of size $\aleph_1$. Let a and b be elements of A which are in distinct equivalence classes. Then $\mathrm{tp}(a) = \mathrm{tp}(b)$. Yet there can be no automorphism carrying a to b, since such an automorphism would have to carry one equivalence class to the other.

Remark. In the expansion $\mathfrak{A}^+$ constructed in the proof of the proposition the relation $a \sim b$ iff $\mathrm{tp}(a) = t(b)$ is definable. This is a very unusual property for a structure.

Question. What does it mean about the theory T if in every model of T (for all $n \in \omega$), the equivalence relation $a \sim b$ iff $\mathrm{tp}(a) = \mathrm{tp}(b)$ is definable?

For example in the theory of equality this is true. Also in the theory of dense linear orders it is true. The condition fails for the theory of the reals as a field.

This question will be answered later in the course.

In the following we need a general lemma about topological groups.

Lemma. *Let be G a topological group and $H \leq G$ a subgroup and $U \leq G$ an open subset. If $U \subseteq H$ then H is open.*

Proof. H is the union of cosets of U, i.e.

$$H = \bigcup_{h \in H} hU$$

and since multiplication by $h \in H$ is a homeomorphism of H it follows that hU is open. Thus H is open. □

In the case where the structures under consideration are countable there is a tighter connection between structure and automorphisms.

Notation. For $b \in A^n$ we shall denote the stabilizer $\mathrm{Aut}(\mathfrak{A})_{(b)}$ by $\mathrm{Aut}(\mathfrak{A}/b)$.

Theorem 1. *Let $\mathfrak{A}$ be a countable τ-structure, and $H \leq \mathrm{Aut}(\mathfrak{A})$ a closed subgroup. The following are equivalent.*

i) H *is open.*
ii) $|\mathrm{Aut}(\mathfrak{A})/H| \leq \aleph_0$.
iii) $|\mathrm{Aut}(\mathfrak{A})/H| < 2^{\aleph_0}$.

Remark. Note that there are at most $2^{\aleph_0}$ elements of $|\mathrm{Aut}(\mathfrak{A})/H|$.

Proof. i) $\Rightarrow$ ii). If H is open then it contains a basic open set, i.e. there exists $a, b \in A^n$ such that $U_{a,b} \subseteq H$. Now as observed last time $U_{a,b}$ is a coset of the stabilizer of b. Since H is a group it must contain the stabilizer itself. Thus

$$\mathrm{Aut}(\mathfrak{A}/b) \leq H.$$

So

$$\mathrm{Aut}(\mathfrak{A}).b \cong \mathrm{Aut}(\mathfrak{A})/\mathrm{Aut}(\mathfrak{A}/b)$$

as $\mathrm{Aut}(\mathfrak{A})$-sets. So

$$|\mathrm{Aut}(\mathfrak{A}).b| = |\mathrm{Aut}(\mathfrak{A})/\mathrm{Aut}(\mathfrak{A}/b)| \geq |\mathrm{Aut}(\mathfrak{A})/H|.$$

But $A^n \supseteq \mathrm{Aut}(\mathfrak{A}).b$ so $|\mathrm{Aut}(\mathfrak{A})/H| \leq |A^n| \leq \aleph_0$.

ii) $\Rightarrow$ iii). Clear.

iii) $\Rightarrow$ i). This step will require some work. We shall prove the contrapositive. We assume that H is not open and use this to show that the index of H in $\mathrm{Aut}(\mathfrak{A})$ has size $2^{\aleph_0}$. We build a tree inside of $\mathrm{Aut}(\mathfrak{A})$ which remains a tree when we mod out by H.

We will construct a sequence $(a_i)_{i\in\omega}$ of finite sequences from A, and a sequence $(\sigma_i)_{i\in\omega}$ from $\mathrm{Aut}(\mathfrak{A})$. For each $T \subseteq \{0, 1, \dots, n-1\}$, say $T = \{i_1 < i_2 < \cdots < i_l\}$ we define

$$\sigma_T := \prod_{i\in T} \sigma_i = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_l}.$$

We shall arrange that the following hold for the sequences $(a_i)_{i\in\omega}$ and $(\sigma_i)_{i\in\omega}$:

- for $i > j$ we have $\sigma_i(a_j) = a_j$.
- for each $n \in \omega$ and $S, T \subseteq \{0, 1, \dots, n-1\}$ if $S \neq T$ then $\sigma_S \not\equiv \sigma_T \pmod{H}$ on $\{a_0, \dots, a_{n-1}\}$.

Remark. If the first condition is satisfied then it does make sense to define σ_T even for infinite $T \subseteq \omega$, as long as we restrict to the a_i's. I.e. the map $\sigma_T : \{a_i\} \to A^*$ defined by $\sigma_T(x) = \prod_{i\in T} \sigma_i(x)$ is well-defined. Since $\mathrm{Aut}(\mathfrak{A})$ is closed in $\mathrm{Sym}(A)$

then for each $T \subseteq \omega$ there is some $\widetilde{\sigma_T} \in \mathrm{Aut}(\mathfrak{A})$ such that $\widetilde{\sigma_T}|_{\{a_i\}} = \sigma_T$. So if we succeed in constructing the sequences (a_i) and (σ_i) to satisfy the two conditions then we can find $2^{\aleph_0}$ many automorphisms which are different mod H.

We define the sequences by induction. The 0 case and the $n+1$ cases are the same. So we just do the $n+1$ step.

Suppose that $(a_i)_{i<n}$ and $(\sigma_i)_{i<n}$ have been obtained (for $n = 0$ this just means the sequences are empty). We look for a_n and σ_n.

By hypothesis H is *not* open. By the lemma H does not contain any open subgroups. In particular H does not contain the stabilizer of the sequence constructed thus far. I.e.

$$\mathrm{Aut}(\mathfrak{A}/(a_0, \ldots, a_{n-1})) \not\subseteq H$$

so there is some $\sigma_n \in \mathrm{Aut}(\mathfrak{A}/(a_0, \ldots, a_{n-1})) \setminus H$.

We now claim that there is some $a_n \in A^m$ such that for all $h \in H$ we have $h(a_n) \neq \sigma_n(a_n)$. This is true since if not, then for all $a, b \in A^m$ (and all $m \in \omega$) if

$$\sigma_n \in U_{a,b}$$

then there is some $h \in U_{a,b} \cap H$ so that σ_n is in the closure of H. But H was assumed closed! So $\sigma_n \in H$, which is a contradiction. So we can find a_n such that σ_n and h disagree on a_n (for all $h \in H$).

By induction we have defined the sequences $(a_i)_{i<\omega}$ and $(\sigma_i)_{i<\omega}$. We must check that they satisfy the two conditions. The first property is clear since $\sigma_n \in \mathrm{Aut}(\mathfrak{A}/(a_0, \ldots, a_{n-1}))$ so σ_n acts trivially on a_i for $i < n$.

To check the second property suppose $S, T \subseteq \{0, \ldots, n\}$ with $S \neq T$. Let j be the first place they differ. With out loss of generality assume $j \in S$.

If $j < n$ then

$$\sigma_S(a_j) = \sigma_{S\setminus\{n\}}(a_j) \quad \text{and} \quad \sigma_T(a_j) = \sigma_{T\setminus\{n\}}(a_j)$$

since σ_n acts on a_j trivially for $j < n$. By induction we may assume $\sigma_{S\setminus\{n\}}$ and $\sigma_{T\setminus\{n\}}$ are inequivalent mod H on the set $\{a_0, \ldots, a_{n-1}\}$.

Now suppose $j = n$. Then $S = T \cup \{n\}$. Suppose there is some $h \in H$ such that $\sigma_S = \sigma_T h$ on $\{a_0, \ldots, a_n\}$. Then

$$\sigma_T h(a_n) = \sigma_S(a_n) = \sigma_T \sigma_n(a_n)$$

so multiplying by σ_T^{-1} we have $h(a_n) = \sigma_n(a_n)$ which is a contradiction with the construction og σ_n.

By the earlier remarks we have now shown that $|\mathrm{Aut}(\mathfrak{A})/H| \geq 2^{\aleph_0}$. Since $\mathfrak{A}$ is countable we must therefore have that $|\mathrm{Aut}(\mathfrak{A})/H| = 2^{\aleph_0}$. □

The above result has a more model-theoretic interpretation which we now develop.

We have seen that closed subgroups of automorphism groups come from expansions. Suppose $\tau \subseteq \tau^+$ is an extension of signatures and $\mathfrak{A}$ is a τ-structure and $\mathfrak{A}^+$ is an expansion. For each τ-automorphism σ of $\mathfrak{A}$ we can find an extension $\mathfrak{A}^{+\sigma}$ of $\mathfrak{A}$ such that $\sigma : \mathfrak{A}^+ \longrightarrow \mathfrak{A}^{+\sigma}$ is an isomorphism of τ^+ structures. To define $\mathfrak{A}^{+\sigma}$ let

- $R^{\mathfrak{A}^{+\sigma}} := \sigma(R^{\mathfrak{A}^+})$ for all $R \in \mathcal{R}_{\tau^+}$.
- $c^{\mathfrak{A}^{+\sigma}} := \sigma(c^{\mathfrak{A}^+})$ for all $c \in \mathcal{C}_{\tau^+}$.
- $f^{\mathfrak{A}^{+\sigma}}(b) := \sigma(f^{\mathfrak{A}^+}(\sigma^{-1}(b)))$ for all $f \in \mathcal{F}_{\tau^+}$.

Conversely if $\widetilde{\mathfrak{A}}$ is an expansion of $\mathfrak{A}$ to τ^+ such that $\mathfrak{A}^+ \cong \widetilde{\mathfrak{A}}$ then there is some $\sigma \in \mathrm{Aut}(\mathfrak{A})$ such that $\widetilde{\mathfrak{A}} = \mathfrak{A}^{+\sigma}$ (just pick an isomorphism $\mathfrak{A}^+ \cong \widetilde{\mathfrak{A}}$).

Moreover $\mathfrak{A}^+ = \mathfrak{A}^{+\sigma}$ if and only if $\sigma \in \mathrm{Aut}(\mathfrak{A}^+)$. So we can identify the set of expansions of $\mathfrak{A}$ which are isomorphic to $\mathfrak{A}^+$, with the set of cosets $\mathrm{Aut}(\mathfrak{A})/\mathrm{Aut}(\mathfrak{A}^+)$.

So now restating the theorem in these terms we get.

Theorem 2. *Let $\tau \subseteq \tau^+$ be an expansion of signatures and $\mathfrak{A}$ a countable τ-structure, and $\mathfrak{A}^+$ and expansion to τ^+. The following are equivalent.*

i) $\mathrm{Aut}(\mathfrak{A}^+)$ *is an open subgroup of* $\mathrm{Aut}(\mathfrak{A})$.
ii) There are at most $\aleph_0$ expansions of $\mathfrak{A}$ which are isomorphic to $\mathfrak{A}^+$.
iii) There are strictly less than $2^{\aleph_0}$ expansions which are isomorphic to $\mathfrak{A}^+$.

furthermore if these conditions are satisfied then there exists $m \in \omega$ and $a \in A^m$ such that

$$\mathrm{Aut}(\mathfrak{A}/a) \subseteq \mathrm{Aut}(\mathfrak{A}^+).$$

As a corollary we get.

Corollary. *For $\mathfrak{A}$ a countable τ-structure the following are equivalent.*

i) $|\mathrm{Aut}(\mathfrak{A})| < \aleph_0$.
ii) $|\mathrm{Aut}(\mathfrak{A})| < 2^{\aleph_0}$.
iii) There is some m and $b \in A^m$ such that $\mathfrak{A}_b$ is rigid, i.e. $\mathrm{Aut}(\mathfrak{A}_b) = \{id\}$.

Proof. We have seen the equivalence between *i)* and *ii)*.

Let $\tau^+ := \tau_A$ and $\mathfrak{A}^+ := \mathfrak{A}_A$, then $\mathrm{Aut}(\mathfrak{A}^+) = \{id\}$. If $|\mathrm{Aut}(\mathfrak{A})| < 2^{\aleph_0}$ then by the theorem $\{id\} \leq \mathrm{Aut}(\mathfrak{A}^+) \leq \mathrm{Aut}(\mathfrak{A})$ is open! So, again by the theorem, there is some $b \in A^m$ such that $\mathrm{Aut}(\mathfrak{A}/b) \leq \mathrm{Aut}(\mathfrak{A}^+) = \{id\}$, os $\mathfrak{A}_b$ is rigid. $\square$