Math 225A – Model Theory

Speirs, Martin

Autumn 2013

General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 4

We carry on where we left off last time.
Let τ be a signature and $\mathscr{L}(\tau)$ be a language. We shall write $\mathscr{L}$ instead of $\mathscr{L}(\tau)$ when the context is clear. Let $\mathfrak{A}$ an $\mathscr{L}$-structure. If T is the collection of all atomic sentences which hold in $\mathfrak{A}$ then T has two properties

- If t is a closed term then $t = t$ is in T.
- If $\varphi(x)$ is atomic and $s = t$ is in T then $\varphi(s) \in T$ if and only if $\varphi(t)$.

Any set of atomic sentences satisfying these two properties is called **=-closed**. We now prove a result about the existence of a model of a theory in much the same way as with the term algebra. We take a theory where we would like to find a model and basically just letting the language serve this goal.

Remark. If S is any set of $\mathscr{L}(\tau)$-sentences then there is a smallest =-closed set $\tilde{S}$ containing S.

So the following proposition could be applied to any set of $\mathscr{L}(\tau)$-sentences by first passing to the =-closure.

Lemma. *If T is an =-closed set of atomic sentences then there exists an $\mathscr{L}$-structure $\mathfrak{A}$ such that T is precisely the set of atomic sentences true of $\mathfrak{A}$, and such that every element of $\mathfrak{A}$ is $t^{\mathfrak{A}}$ for some closed term t.*

Remark. If $T = \emptyset$ then $\mathfrak{A}$ will be the term algebra.

Proof. The domain of $\mathfrak{A}$ will be $\mathscr{T}(\tau)$ (closed $\mathscr{L}(\tau)$-terms) modulo the equivalence relation given by $s \sim t$ if and only if $s = t \in T$. Let us show that this is indeed an equivalence relation.

- *Reflexivity*: For all t we have $t \sim t$ since by assumption $t = t \in T$.

- *Symmetry*: Suppose $s \sim t$ so that $s = t \in T$. Consider the formula $\varphi(x)$ given by $x = s$. φ is an atomic formula with one free variable, x. Now $\varphi(s)$ is in T and so by $=$-closure of T we have $\varphi(t)$ in T, i.e. $t = s \in T$ and so $t \sim s$.
- *Transitivity*: Suppose $s \sim t$ and $t \sim r$. Let $\varphi(x)$ be $s = x$. Then $\varphi(t) \in T$ and so by the $=$-closure of T we have that $\varphi(r) \in T$ so $s \sim r$.

Thus, $\sim$ is an equivalence relation. We now let the domain of $\mathfrak{A}$ be $A := \mathscr{T}(\tau)/\sim$, and denote the equivalence class containing t by $[t]_\sim$. To define the τ-structure on $\mathfrak{A}$ we set

- for $c \in \mathcal{C}_\tau$, $c^{\mathfrak{A}} = [c]_\sim$
- for $f \in \mathcal{F}_\tau$ of arity n we define $f^{\mathfrak{A}}([t_0]_\sim, \ldots, [t_{n-1}]_\sim) = [f(t_0, \ldots, t_{n-1})]_\sim$
- for $R \in \mathcal{R}_\tau$ of arity n then $([t_0]_\sim, \ldots, [t_{n-1}]_\sim) \in R^{\mathfrak{A}}$ if and only if $R(t_0, \ldots, t_{n-1}) \in T$.

We must show that these definitions are well-defined and that $\mathfrak{A}$ has the desired properties. For constants there is no problem. But for an n-ary function symbol $f \in \mathcal{F}_\tau$ we must show that the value af $f^{\mathfrak{A}}$ does not depend on the choice of representatives. The same goes for relation symbols. Suppose $(t_0, \ldots, t_{n-1})$ is a sequence of terms which are equivalent, coordinate-wise, to $(s_0, \ldots, s_{n-1})$. Then we must show that $f^{\mathfrak{A}}([t_0]_\sim, \ldots, [t_{n-1}]_\sim) = f^{\mathfrak{A}}([s_0]_\sim, \ldots, [s_{n-1}]_\sim)$. To see this we use that

$$f(s_0, \ldots, s_{n-1}) = f(s_0, \ldots, s_{n-1}) \in T$$

and that T is $=$-closed so since $s_0 \sim t_0$ we have

$$f(t_0, s_1, \ldots, s_{n-1}) = f(s_0, \ldots, s_{n-1}) \in T$$

and applying this n-times we get

$$f(t_0, \ldots, t_{n-1}) = f(s_0, \ldots, s_{n-1}) \in T.$$

Similarly, suppose $R \in \mathcal{R}_\tau$ is an n-ary relation symbol, and suppose $R^{\mathfrak{A}}([s_0]_\sim, \ldots, [s_{n-1}]_\sim)$ then by succesively substituting t_i's for s_i's we will see that $R^{\mathfrak{A}}([t_0]_\sim, \ldots, [t_{n-1}]_\sim)$ also holds. So $\mathfrak{A}$ is now an $\mathscr{L}$-structure.

To show that T is exactly the set of atomic sentences that are satisfied by $\mathfrak{A}$ we use induction on the complexity of atomic sentences. For the case $t = s$ we have

$$\begin{aligned} \mathfrak{A} \models s = t \quad & \text{iff} \quad s^{\mathfrak{A}} = t^{\mathfrak{A}} \\ & \text{iff} \quad [s]_\sim = [t]_\sim \\ & \text{iff} \quad s = t \in T \end{aligned}$$

and similarly, for the case $R(t_0, \dots, t_{n-1})$ we have

$$\begin{aligned} \mathfrak{A} \models R(t_0, \dots, t_{n-1}) \quad & \text{iff} \quad R^{\mathfrak{A}}(t_0^{\mathfrak{A}}, \dots, t_{n-1}^{\mathfrak{A}}) \\ & \text{iff} \quad R(t_0, \dots, t_{n-1}) \in T. \end{aligned}$$

Now for the final claim, that all elements of $\mathfrak{A}$ have the form $t^{\mathfrak{A}}$ one uses induction on the complexity of terms to show that $[t]_\sim = t^{\mathfrak{A}}$. This is clear from the above construction. □

Proposition. *Let T be an arbitrary set of atomic sentences. Then there is a structure $\mathfrak{A}$ such that*

1. $\mathfrak{A} \models T$
2. *Every $x \in \mathrm{dom}(\mathfrak{A})$ is of the form $t^{\mathfrak{A}}$ for some $\mathscr{L}$-term.*
3. *If $\mathfrak{B} \models T$ then there is a unique homomorphism $f : \mathfrak{A} \to \mathfrak{B}$.*

Proof. For *(1.)* and *(2.)* take the =-closure of T and apply the above lemma. *(3.)* follows from the diagram lemma proved last time. □

By *(3.)* of the proposition, $\mathfrak{A}$ is an initial object in the category of models of T.

Example. If $T = \emptyset$ and $\tau = \{f\}$ is a binary function, then we cannot form any closed terms and so cannot form any sentences.

Example. If F is a field and $p(x) \in F[x]$ is an irreducible polynomial over F then considering $F[x]$ as an $\mathscr{L}(\tau_{rings} \cup \{c_i\}_{i \in F[x]})$-structure take

$$T = \{ \text{ equations true in } F[x]\}.$$

Then $F[x]$ is the initial structure in the category of T-models. Now consider the enlarged collection $T \cup \{p(x) = 0\}$ and take the =-closure. The initial model for this collection will yield the ring $\mathfrak{A} = F[x]/(p(x))$ where we have added a root to $p(x)$. Moreover we get the quotient map $F[x] \to F[x]/(p(x))$.

Relations defined by atomic formulae

Given an $\mathscr{L}$-structure $\mathfrak{A}$ with domain A, and $\varphi(x_0, \dots, x_{n-1})$ an atomic $\mathscr{L}$-formula we define $\varphi(A^n)$ to be $\{\bar{a} \in A^n : \mathfrak{A} \models \varphi(\bar{a})\}$. We can also allow parameters; if $\psi(\bar{x}, \bar{y})$ is an atomic formula and $\bar{b} \in A^m$ then

$$\psi(A^n, \bar{b}) = \{\bar{a} \in A^n : \mathfrak{A} \models \psi(\bar{a}, \bar{b})\}.$$

Infinitary languages

Given a signature τ we now define the *infinitary* language $\mathscr{L}_{\infty\omega}$ associated to τ. Roughly speaking the two subscripts describe how many conjunction/disjuntions we are allowed to use and how many quantifications we are allow. The first subscript '∞' indicates that we will allow infinitely many conjunctions and disjunctions. The second subscript 'ω' indicates that we will allow only finitely many quantifiers in a row.

The symbols of $\mathscr{L}_{\infty\omega}$ are all symbols from the signature τ together with the usual logical symbols:

$$=, \neg, \bigwedge, \bigvee, \forall, \exists.$$

The terms, atomic formulas, and literals are defined in the same way as before (i.e. for first-order logic).

Definition. $\mathscr{L}_{\infty\omega}$ is the smallest class such that

- all atomic formulas are in $\mathscr{L}_{\infty\omega}$
- if $\varphi \in \mathscr{L}_{\infty\omega}$ then $\neg\varphi \in \mathscr{L}_{\infty\omega}$
- if $\Phi \subseteq \mathscr{L}_{\infty\omega}$ then $\bigvee \Phi$ and $\bigwedge \Phi$ are in $\mathscr{L}_{\infty\omega}$
- if $\varphi \in \mathscr{L}_{\infty\omega}$ then $\forall x\varphi$ and $\exists x\varphi$ are in $\mathscr{L}_{\infty\omega}$

Remark. We are allowing $\Phi \subseteq \mathscr{L}_{\infty\omega}$ to be an *arbitrary* subset, so we are allowing arbitrary conjunctions and disjunctions, contrary to the case for the usual first-order logic.

Given an $\mathscr{L}$-structure $\mathfrak{A}$ (with domain A) we can now extend the notion of satisfaction "$\models$" to arbitrary formulas of $\mathscr{L}_{\infty\omega}$;

- For atomic formulas the $\models$ relation is the same as before.
- Given $\varphi(\bar{x}) \in \mathscr{L}_{\infty\omega}$ then $\mathfrak{A} \models \neg\varphi(\bar{a})$ if and only if it is not the case that $\mathfrak{A} \models \varphi(\bar{a})$.
- Given $\Phi(\bar{x}) \subseteq \mathscr{L}_{\infty\omega}$ then $\mathfrak{A} \models \bigwedge \Phi(\bar{a})$ if and only if, for all $\varphi(\bar{x}) \in \Phi(\bar{x})$ $\mathfrak{A} \models \varphi(\bar{a})$.
- Given $\Phi(\bar{x}) \subseteq \mathscr{L}_{\infty\omega}$ then $\mathfrak{A} \models \bigvee \Phi(\bar{a})$ if and only if, for at least one of $\varphi(\bar{x}) \in \Phi(\bar{x})$ we have $\mathfrak{A} \models \varphi(\bar{a})$.
- Given $\varphi(y, \bar{x}) \in \mathscr{L}_{\infty\omega}$, then $\mathfrak{A} \models \forall y\varphi(y, \bar{a})$ if and only if for all $b \in A$ we have $\mathfrak{A} \models \varphi(b, \bar{a})$.
- Given $\varphi(y, \bar{x}) \in \mathscr{L}_{\infty\omega}$, then $\mathfrak{A} \models \exists y\varphi(y, \bar{a})$ if and only if for at least one $b \in A$ we have $\mathfrak{A} \models \varphi(b, \bar{a})$.

Now we say that **first-order logic** is the language $\mathscr{L}_{\omega\omega}$ where we allow only finite subsets Φ (in other words we have only finite conjunctions and disjunctions), and only finitely many quantifiers. In general for some cardinal κ we get a language $\mathscr{L}_{\kappa\omega}$ where we allow the subsets $\Phi \subseteq \mathscr{L}_{\kappa\omega}$ to have size $< \kappa$.

In model theory we most often either work within $\mathscr{L}_{\omega\omega}$ or with $\mathscr{L}_{\omega_1\omega}$. The latter language allows *countably* many conjunctions and disjunctions. There are however several properties of first-order logic that the infinitary logics fail to have. Some of these are demonstrated by the following suggested exercises.

Exercise 1. Give an example of an $\mathscr{L}_{\omega_1\omega}$ sentence Φ such that every finite subsentence og Φ is satisfiable, but Φ is not. (So compactness fails).

Exercise 2. Axiomatize the following classes of structures with some single sentence in some language using $\mathscr{L}_{\omega_1\omega}$:

- Torsion-free abelian groups.
- Finitely generated fields.
- Linear orders isomorphic to $(\mathbb{Z}, <)$.
- Connected graphs.
- Finite valence graphs.
- Cycle-free graphs.

Exercise 3. Give an example of a countable language $\mathscr{L}$ and an $\mathscr{L}_{\omega_1\omega}$ sentence Φ such that every models of Φ has cardinality at least $2^{\aleph_0}$. (So Downward Løwenheim-Skolem fails).

Axiomatization

Definition. A class of $\mathscr{L}$-structures $\mathcal{K}$ is **axiomatizable** if there is some $\mathscr{L}$-theory T such that the class of $\mathscr{L}$-structures satisfying T is $\mathcal{K}$. $\mathcal{K}$ is **$\mathscr{L}$-definable** if we can take $T = \{\varphi\}$ for some $\mathscr{L}$-sentence φ.

The following lemma is important.

Lemma. *Let $\mathfrak{A}$ be an $\mathscr{L}$-structure and $X \subseteq \mathrm{dom}(\mathfrak{A})$ and Y some relation defined by a formula with parameters from X. Then if $f \in \mathrm{Aut}(\mathfrak{A})$ (the group of $\mathscr{L}$-structure automorphisms of $\mathfrak{A}$) fixes X point-wise then f fixes Y set-wise (i.e. $f(Y) = Y$).*

In other words definable sets are invariant under those automorphisms which fix the parameter space. For instance if a set Y is definable without parameters then $Y = f(Y)$ for *every* automorphism. This puts restrictions on the definable sets.

The Arithmetical Hierarchy

The theory of arithmetic is the theory of the structure $\mathbb{N} = (\omega, 0, 1, +, \cdot, <)$.

Definition. Let $\exists x < y\ \varphi$ and $\forall x < y\ \varphi$ be abbreviations of the formulas $\exists x(x < y \wedge \varphi)$ and $\forall x(x < y \to \varphi)$ respectively. These are called **bounded quantifiers**.

Definition. The arithmetic hierarchy is the following hierarchy of subsets of ω.

- φ is Σ^0_0 and Π^0_0 if all quantifiers are bounded.
- φ is in Σ^0_{n+1} if $\varphi = \exists \bar{x} \psi$ for some ψ in Π^0_n.
- φ is in Π^0_{n+1} if $\varphi = \forall \bar{x} \psi$ where ψ is in Σ^0_n.

So the subscript of Σ^0_n and Π^0_n is the number of alterations of (unbounded) quantifiers appearing in the formulae. It can in fact be shown that this hierarchy is proper, i.e. the inclusions $\Sigma^0_n \subseteq \Sigma^0_{n+1}$ are proper for all $n \in \omega$.