Math 225A – Model Theory

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Autumn 2013

General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 20

Heirs and Coheirs

Notation. Recall that $S_n(A)$ denotes the space of n-types over A. The union $\bigcup_{n=1}^{\infty} S_n(A)$ is written simply as $S(A)$.

Definition. Let $\mathfrak{M}$ be a τ-structure and $A \subseteq B \subseteq \operatorname{dom}(\mathfrak{M})$. Given $p \in S(A)$ and $q \in S(B)$ with $p \subseteq q$ (as sets), then q is an **heir** of p if, for each formula $\varphi(x, y) \in \mathscr{L}(\tau_A)$, if there exists b from B such that $\varphi(x, b) \in q$, then there exists some b' from A such that $\varphi(x, b') \in p$.

Roughly, q is an heir of p if every formula represented in q is already represented in p.

A related but different notion is that of coheir.

Definition. Given p and q as above, we say that q is a **coheir** of p if, for each formula $\theta(x)$ from q there exists $a \in A$ such that $\mathfrak{M} \models \theta(a)$.

Remark (equivalent definition of coheir). q is a coheir of p if, for each formula $\varphi(x, y) \in \mathscr{L}(\tau_A)$ if $b \in B$ and $\varphi(x, b) \in q$ then there is some $a \in A$ such that $\mathfrak{M} \models \varphi(a, b)$.

Notation. Given a set A and a tuple b we denote by Ab the set $A \cup \{b_i : b_i \text{ from } b\}$.

Remark (In what sense is the notion of *coheir* "co" to the notion of *heir*?). Given $A \subseteq \operatorname{dom}(\mathfrak{M})$ and a, b tuples, then

$$\operatorname{tp}(a/Ab) \text{ is an heir of } \operatorname{tp}(a/A)$$

if and only if

$$\operatorname{tp}(b/Aa) \text{ is a coheir of } \operatorname{tp}(b/A).$$

Proof.

- To say that $\mathrm{tp}(a/Ab)$ is an heir of $\mathrm{tp}(a/A)$ is to say that for each $\varphi(x,y) \in \mathscr{L}(\tau_A)$ with $\varphi(x,b) \in \mathrm{tp}(a/Ab)$ there exists b' from A such that $\varphi(x,b') \in \mathrm{tp}(a/A)$,
- which is to say that for each $\varphi(x,y) \in \mathscr{L}(\tau_A)$ we have, $\mathfrak{M} \models \varphi(a,b)$ if and only if there is some b' from A such that $\mathfrak{M} \models \varphi(a,b')$,
- which is to say that, for each $\varphi(x,y) \in \mathscr{L}(\tau_A)$ then $\varphi(a,y) \in \mathrm{tp}(b/Ab)$ if and only if there exists b' from A such that $\mathfrak{M} \models \varphi(a,b')$,
- which is to say that $\mathrm{tp}(b/Ab)$ is a coheir of $\mathrm{tp}(b/A)$.

□

We will now show that heirs and coheirs always exist (if A is the domain of a model).

Proposition. *If $\mathfrak{M} \preccurlyeq \mathfrak{N}$ and $B \supseteq M$ and $p \in S(M)$ then there exist $q, r \in S(B)$ such that q is an heir of p and r is a coheir of p.*

Proof. We shall write down "what we don't want". To get q we want an extension of p such that no formulas are represented which are not already represented by p. Thus consider the theory

$$Q := p \ \cup \ \{\neg\theta(x,b) : \theta(x,y) \in \mathscr{L}(\tau_M) \text{ not represented in } p, \text{ and with } b \text{ from } B\}$$

We claim: *Q is consistent.*

Proof. (of claim) If not then there exists a finite list $\theta_1(x,b_1), \dots, \theta_n(x,b_n)$ of formulae where $\theta_i(x,y)$ is not represented in p, and some formula $\varphi(x) \in p$ such that

$$\vdash \varphi(x) \longrightarrow \bigvee_i \theta_i(x,b_i).$$

In particular, by generalization,

$$\mathfrak{N}_B \models \exists y_1, \dots, y_n \forall x \left(\varphi(x) \longrightarrow \bigvee_i \theta_i(x,y_i) \right).$$

But now $\exists y_1, \dots, y_n \forall x\, (\varphi(x) \longrightarrow \bigvee_i \theta_i(x,y_i))$ is a τ_A sentence, so since $\mathfrak{M} \preccurlyeq \mathfrak{N}$ we have that $\mathfrak{M}_B$ also satisfies this sentence. Thus there exists $b_1', \dots, b_n'$ from M such that

$$\mathfrak{M}_B \models \forall x \left(\varphi(x) \longrightarrow \bigvee_i \theta_i(x,y_i) \right).$$

Since each $\theta_i(x,b_i')$ is a τ_M-formula, either $\theta_i(x,b_i') \in p$ or $\neg\theta_i(x,b_i') \in p$ (since p is complete). Now since $\varphi \in p$ there must indeed be some i such that $\theta_i(x,b_i') \in p$. But this means that $\theta_i(x,y)$ is represented in p, which is a contradiction. □

Now by compactness Q is consistent. Letting $(\mathfrak{M}', a) \models Q$ (where a is meant to be substituted into the tuple x in the definition of Q) and $q := \mathrm{tp}(a/B)$ we have $q \supseteq p$ and q is an heir of p.

Now to get the coheir, consider

$$R := p \cup \{\neg\theta(x, b) : \theta(x, y) \in \mathscr{L}(\tau_M) \text{ and there does not exist } a' \text{ from } M \text{ s.t. } \mathfrak{M} \models \theta(a', b)\}.$$

Once again we claim: *R is consistent.*

Proof. (of claim) If not, then there exists some $\varphi(x) \in p$ and $\theta_1(x, b_1), \ldots, \theta_n(x, b_n)$ such that

$$\vdash \varphi(x) \longrightarrow \bigvee_i \theta_i(x, b_i)$$

and such that each $\theta_i(x, b_i)$ is *not* realized in $\mathfrak{M}$.

Since φ is in p and p extends $\mathrm{Th}(\mathfrak{M}_M)$ and p is consistent, so

$$\mathfrak{M}_M \models \exists x \varphi(x)$$

so let a be a witness, i.e. $\mathfrak{M}_M \models \varphi(a)$. Since $\mathfrak{M} \preccurlyeq \mathfrak{N}$ and since

$$\vdash \varphi(x) \longrightarrow \bigvee_i \theta_i(x, b_i)$$

it follows that there is some i such that $\mathfrak{N} \models \theta_i(a, b_i)$, i.e. $\theta_i(x, b_i)$ is realized in $\mathfrak{M}$, which is a contradiction, thus the claim holds. □

Let $(\mathfrak{M}', a) \models R$. Then $r = \mathrm{tp}(a/B)$ is the desired coheir for p. □

So given p as in the theorem, we can extend it to get an heir (q) and a coheir (r).

Example. Let $\mathfrak{M} = ((0, 1), <)$ thought of as an elementary substructure of $\mathfrak{B} = (\mathbb{R}, <)$ and let p be the type generated by the formula $\{x > a : a \in (0, 1)\}$. What do the heirs and coheirs of p look like?

Suppose $q \in S(\mathbb{R})$ is an heir of p, and suppose $r \in S(\mathbb{R})$ is a coheir of p. Is $x > 2$ in r or q?

It cannot be in r since $x > 2$ then there would have to be some $a \in (0, 1)$ such that $a > 2$ was satisfied by $\mathfrak{B}$. Of course there isn't. In general r is (generated by) $\{a < x : a < 1\}$.

Now $x > 2$ *is* in q since the formula $\psi(x, y) = \neg(x > y)$ is *not* represented in p. In general q is (generated by) $\{x > a : a \in \mathbb{R}\}$.

Note that the heirs and coheirs of p are different.

Example. In many cases it is in fact the case that there are at most two distinct coheirs. We can modify the above example slightly to get an example where p has two coheirs.

Let $\tilde{p} \in S(\mathbb{R})$ be $\{x > a : a \leq \frac{1}{2}\} \cup \{x < a : a > \frac{1}{2}\}$. Which says of x that is is "infinitesimally greater than $\frac{1}{2}$". Then considering $\mathbb{R}$ as an elementary substructure of a model, $\mathfrak{M}$, which has infinitesimals, then there are two coheirs

$$q^{+} = \{x > a : a \leq \frac{1}{2}, a \in M\} \cup \{x < a : a \geq \frac{1}{2}\}$$

and

$$q^{-} = \{x > a : a \in M \text{ such that } \forall r \in \mathbb{R} \text{ if } r > \frac{1}{2}, \text{ then } a < r\}.$$

Preservation Theorems

Earlier in the course we observed that theories which have certain simple syntactic characterizations are also preserved under certain semantic operations. For instance if T admits a universal (i.e. $\forall_1$) axiomatization, then the class of models of T is closed under substructures. We now turn to proving converses of these statements.

Theorem 1. *(Łos-Tarski) If* T *is a theory in some language* $\mathscr{L}(\tau)$ *then the following are equivalent.*

1) *If* $\mathfrak{A} \subseteq \mathfrak{B}$ *with* $\mathfrak{B} \models \mathrm{T}$*, then* $\mathfrak{A} \models \mathrm{T}$.
2) *There is a set* U *of universal sentences such that* T *and* U *have exactly the same models.*

We have already seen that the second condition implies the first. We shall prove the converse implication shortly.

Notation. (As in Hodges) Given τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, we write $\mathfrak{A} \Rrightarrow_{\Delta} \mathfrak{B}$, for a set of τ-sentences, if for all $\delta \in \Delta$ we have

$$\mathfrak{A} \models \delta \qquad \Longrightarrow \qquad \mathfrak{B} \models \delta$$

The case where Δ is the set of existential (i.e. $\exists_1$) sentences is written $\mathfrak{A} \Rrightarrow_{\exists} \mathfrak{B}$.

We now prove that if $\mathfrak{A} \Rrightarrow_{\exists} \mathfrak{B}$ then $\mathfrak{A}$ is a substructure of (an elementary extension of) $\mathfrak{B}$.

Notation. In the following we shall use the notation Diag($\mathfrak{A}$), for the atomic diagram. Unlike the old notion (diag($\mathfrak{A}$)) this will contain all quantifier-free formulas, i.e. it is closed under conjunction and disjunction. This slightly expands the notion used previously, but no extra information is used.

Proposition. *The following are equivalent.*

1) $\mathfrak{A} \Rrightarrow_{\exists} \mathfrak{B}$.

2) *There exists* $\mathfrak{C}$ *such that* $\mathfrak{B} \preccurlyeq \mathfrak{C}$ *and* $\mathfrak{A}$ *embeds into* $\mathfrak{C}$.

Proof. 2) $\Rightarrow$ 1). We have $\mathfrak{C} \equiv \mathfrak{B}$, so in particular, $\mathrm{Th}_{\forall}(\mathfrak{C}) = \mathrm{Th}_{\forall}(\mathfrak{B})$. We know that if $\mathfrak{A}$ is (isomorphic to) a substructure of $\mathfrak{C}$ then $\mathrm{Th}_{\forall}(\mathfrak{C}) \subseteq \mathrm{Th}_{\forall}(\mathfrak{A})$. In other words

$$\mathrm{Th}_{\exists}(\mathfrak{A}) \subseteq \mathrm{Th}_{\exists}(\mathfrak{C}) = \mathrm{Th}_{\exists}(\mathfrak{B})$$

which is to say $\mathfrak{A} \Rrightarrow_{\exists} \mathfrak{B}$.

1) $\Rightarrow$ 2). Consider the theory

$$\mathrm{T} := \mathrm{eldiag}(\mathfrak{B}) \cup \mathrm{diag}(\mathfrak{A})$$

(making sure that the new constant symbols for A and B don't overlap). We claim that T is consistent. If not then, by compactness, there exists some $\varphi(b) \in \mathrm{eldiag}(\mathfrak{B})$ and $\psi(a) \in \mathrm{diag}(\mathfrak{A})$ such that b and a are new constant symbols and such that

$$\vdash \varphi(b) \longrightarrow \neg\psi(a).$$

In particular

$$\mathfrak{B} \models \forall x[\varphi(x) \longrightarrow \neg\psi(x)].$$

Since $\varphi(b) \in \mathrm{eldiag}(\mathfrak{B})$ the above implication shows that $\mathfrak{B} \models \forall x \neg\psi(x)$. But $\mathfrak{A} \Rrightarrow_{\exists} \mathfrak{B}$ and $\mathfrak{A} \models \exists x \psi(x)$, which is a contradiction. Thus T is consistent. Letting $\mathfrak{C} \models \mathrm{T}$, we have that $\mathfrak{C}|_{\tau}$ is the desired structure. □

Example. In the proof above we used the fact that $\mathrm{Th}_{\forall}(\mathfrak{C}) \subseteq \mathrm{Th}_{\forall}(\mathfrak{A})$ whenever $\mathfrak{A}$ is a substructure of $\mathfrak{C}$. We give an example where the containment is strict. Consider $\mathbb{Z} \subseteq \mathbb{R}$ where they are considered τ-structures for $\tau = \{<, 0, 1\}$. Then $\mathbb{Z}$ satisfies $\forall x[x = 1 \vee x = 0 \vee x < 0 \vee x > 1]$ which $\mathbb{R}$ does not.

We can now prove Theorem 1.

Proof. (of Łos-Tarski, Theorem 1) We have already seen 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 1). Let $U := \mathrm{T}_{\forall}$ the set of all universal consequences of T. We must show that $\mathrm{Mod}(\mathrm{T}) = \mathrm{Mod}(U)$. If $\mathfrak{A} \models \mathrm{T}$ then clearly $\mathfrak{A} \models U$. So $\mathrm{Mod}(\mathrm{T}) \subseteq \mathrm{Mod}(U)$.

Now suppose $\mathfrak{A} \models U$. Let $S := \mathrm{T} \cup \mathrm{Diag}(\mathfrak{A})$. We claim that S is consistent. If not then there exists $\varphi(a) \in \mathrm{Diag}(\mathfrak{A})$ such that $\mathrm{T} \vdash \neg\varphi(a)$. Thus $\mathrm{T} \vdash \forall x \neg\varphi(x)$ (since a was a new variable). Since φ is quantifier-free we now see that $\forall x \neg\varphi(x) \in \mathrm{T}_{\forall}$. But sine $\mathfrak{A} \models \mathrm{T}_{\forall}$ and since $\mathfrak{A} \models \exists x \varphi(x)$ we have a contradiction. So S is indeed consistent. Let $\mathfrak{C}$ be a model of S. Then $\mathfrak{A}$ embeds into $\mathfrak{C}$ and we have $\mathfrak{C} \models \mathrm{T}$. By condition 1) we have that $\mathfrak{A} \models \mathrm{T}$. This completes the proof. □

There are many similar kinds of preservation theorems for different types of syntactic classes. Many examples can be found in Chang and Keisler's book.