Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 19

Amalgamation

Amalgamations are useful for realizing many types all at once inside one structure.

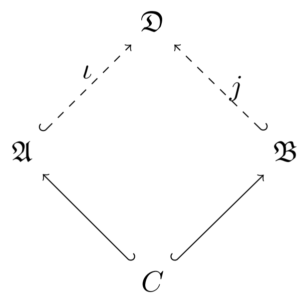
We will prove the Elementary Amalgamation Theorem as a consequence of the Compactness Theorem. Let τ be some signature and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ and $C \subseteq A$ and $C' \subseteq B$ together with a bijection $f : C \to C'$.

Notation. By $\mathfrak{A}_C \equiv \mathfrak{B}_{C'}$ we mean the usual expect that whenever some constant symbol $c \in C$ is used in a formula on the $\mathfrak{A}$-side then the corresponding constant symbol $f(c) \in C'$ is used on the $\mathfrak{B}$-side.

Theorem 1 (Elementary Amalgamation)**.** *With τ, $\mathfrak{A}$ and $\mathfrak{B}$, C and f as above, if $\mathfrak{A}_C \equiv \mathfrak{B}_{C'}$ then there exists a τ-structure $\mathfrak{D}$ such that*

- *There are elementary embeddings $\iota : \mathfrak{A} \to \mathfrak{D}$ and $j : \mathfrak{B} \to \mathfrak{D}$, such that $\iota|_C = j \circ f$*

I.e. that the following diagram commutes:



Proof. First, without loss of generality we may assume that $A \cap B = C$, as sets, by identifying C and C' via f and then by replacing A and B with new disjoint copies whose only overlap is C.

We aim to use the elementary diagram lemma. Consider the $\mathscr{L}(\tau_{A\cup B})$-theory

$$\mathrm{T} := \mathrm{eldiag}(\mathfrak{A}) \cup \mathrm{eldiag}(\mathfrak{B}).$$

A model of T would then (upon reduction to τ) give us what we want.

Suppose T does not have a model. By compactness there is some finite inconsistent subset of T. I.e. we would have

$$\psi_1(a^{(1)}), \ldots, \psi_n(a^{(n)}) \in \mathrm{eldiag}(\mathfrak{A}) \qquad \text{and} \qquad \varphi_1(b^{(1)}), \ldots, \varphi_m(b^{(m)}) \in \mathrm{eldiag}(\mathfrak{B})$$

where $\psi_i, \varphi_i \in \mathscr{L}(\tau_C)$ and where $a^{(i)}, b^{(i)}$ are tuples from A and B respectively, such that

$$\{\psi_1(a^{(1)}), \ldots, \psi_n(a^{(n)}), \varphi_1(b^{(1)}), \ldots, \varphi_m(b^{(m)})\}$$

is inconsistent.

We now make some adjustments to make things more maneageble.

- We may assume no constant symbols from C appear as coordinates of $a^{(i)}$ or $b^{(i)}$ since if they did then we could absorb them into the formulas ψ_i or φ_i from $\mathscr{L}(\tau_C)$.
- We may assume that $a^{(i)} = a^{(j)} =: a$ and $b^{(i)} = b^{(j)} =: b$ for all i, j, by padding the ψ_i's and φ_i's with dummy variables.
- We may assume that $n = m = 1$, i.e. that there is only one φ_i and ψ_i. This is because the elementary diagram is closed under conjunctions, so letting

$$\varphi := \bigwedge \varphi_i \qquad \text{and} \qquad \psi := \bigwedge \psi_i$$

amounts to the same thing.

So after these reductions we now have that the $\mathscr{L}(\tau_C)$-theory

$$\{\psi(a), \varphi(b)\}$$

is inconsistent (note that $\psi(a) \in \mathrm{eldiag}(\mathfrak{A})$ and $\varphi(b) \in \mathrm{eldiag}(\mathfrak{B})$). Thus we have

$$\models \psi(a) \longrightarrow \neg\varphi(b).$$

Now $\psi(a) \in \mathrm{eldiag}(\mathfrak{A})$ so $\mathfrak{A}_A \models \psi(a)$. Then for any choice of expansion of $\mathfrak{A}_A$ to $\tau_{A\cup\{b'\}}$ we must have

$$\mathfrak{A}_{A,b'} \models \neg\varphi(b').$$

(Here we have used that A and B are disjoint apart from C.) Since this holds for *any* way we interpret b' in $\mathfrak{A}$, it follows that

$$\mathfrak{A}_A \models \forall y \neg\varphi(y).$$

Now $\forall y \neg\varphi(y)$ is a τ_C-sentence. So $\forall y \neg\varphi(y) \in \mathrm{Th}(\mathfrak{A}_C)$. But by assumption $\mathfrak{A}_C \equiv \mathfrak{B}_C$ and so $\mathrm{Th}(\mathfrak{A}_C) = \mathrm{Th}(B_C)$, so that

$$\mathfrak{B} \models \forall y \neg\varphi(y).$$

But $\mathfrak{B}_B \models \varphi(b)$ and so $\mathfrak{B} \models \exists y \varphi(y)$, which is a contradiction.

Therefore T is consistent. Letting $\mathfrak{D}^+$ be a model of T we get the desired $\mathfrak{D}$ as $\mathfrak{D}^+|_\tau$. □

Thus, given a common subset (or even substructure) and two extensions which from their first-order theories look the same relative to the common subset, then they can be amalgamated into a common elementary extension.

Example. Let $\tau = \tau_{abeliangroup}$, $\mathfrak{A} = (\mathbb{R}, +, -, 0)$ and $\mathfrak{B} = (\mathbb{Q}, +, -, 0)$ and $C = \{1\}$. Then we are in the case of the theorem, i.e. $\mathfrak{A}_C \equiv \mathfrak{B}_C$. So we can amalgamate $\mathfrak{A}$ and $\mathfrak{B}$. Now the result of this would be $\mathfrak{B}$. But then $A \cap B = \mathbb{Q}$ would strictly contain C. In fact there is no way of avoiding this.

This example shows that there can be some obstruction which prevents us from amalgamating freely over C, i.e. such that images of A and B inside D have too big an overlap. Another example may illuminate the problem.

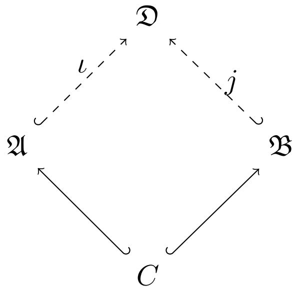
Example. Let $\tau = \tau_{fields}$, $\mathfrak{A} = (\mathbb{Q}(t)^{alg}, +, \cdot, 0, 1)$ and $\mathfrak{B} = (\mathbb{C}, +, \cdot, 0, 1)$ and $C = \mathbb{Q}$. Once again in the amalgamation $\mathfrak{D}$, of $\mathfrak{A}$ and $\mathfrak{B}$ over C, the sets A and B (inside D) will strictly contain C. For instance $\sqrt[3]{5}$ will have to be in this intersection.

Definition. Let $\mathfrak{A}$ be a τ-structure and $X \subseteq \mathrm{dom}(\mathfrak{A})$. An element $a \in \mathrm{dom}(\mathfrak{A})$ is **algebraic over** X if there is a first-order formula $\varphi(x, \bar{y}) \in \mathscr{L}(\tau)$ and a tuple $\bar{b}$ from X such that $\mathfrak{A} \models \varphi(a, \bar{b}) \wedge \exists^{\leq n} x \varphi(x, \bar{b})$ for some $n \in \omega$. We write $\mathrm{acl}_{\mathfrak{A}}(X)$ for the set of elements of $\mathfrak{A}$ that are algebraic over X. We say that X is **algebraically closed** if $\mathrm{acl}_{\mathfrak{A}}(X) = X$. Another way of stating this is that whenever φ is a formula with parameters from X and $\varphi(\mathfrak{A})$ is finite, then $\varphi(\mathfrak{A}) \subseteq X$.

Theorem 2. *Let $\mathfrak{A}$, $\mathfrak{B}$ and C be as in the previous theorem (where we have identified C and C'). Assume further that C is algebraically closed as a subset of $\mathfrak{A}$. Then there is a τ-structure $\mathfrak{D}$ such that*

- *There are elementary embeddings $\iota : \mathfrak{A} \to \mathfrak{D}$ and $j : \mathfrak{B} \to \mathfrak{D}$, such that $\iota|_C = j \circ f$*
- *$\iota(A) \cap j(B) = \iota(C)$.*

I.e. we have the following commutative diagram



where $\iota(A) \cap j(B) = \iota(C)$.

Proof. We follow much the same proof as the Elementary Amalgamation Theorem above. Therefore this will only be a sketch, to show the main differences between the two proofs.

As before we assume $A \cap B = C$.

Now let

$$\mathrm{T} := \mathrm{eldiag}(\mathfrak{A}) \cup \mathrm{eldiag}(\mathfrak{B}) \cup \{a \neq b : a \in A \setminus C \text{ and } b \in B \setminus C\}.$$

We aim to show that T is consistent, this will clearly suffice for the theorem. As before, if T is not consistent then we get $\psi(\bar{a})$ and $\varphi(\bar{b})$ such that $\psi, \varphi \in \mathscr{L}(\tau_C)$ and $\psi(\bar{a}) \in \mathrm{eldiag}(\mathfrak{A})$ and $\varphi(\bar{b}) \in \mathrm{eldiag}(\mathfrak{B})$, *where all coordinates of* $\bar{a}$ *and* $\bar{b}$ *are not from* C. Furthermore we now also know that

$$\{\psi(\bar{a})\} \cup \{\varphi(\bar{b})\} \cup \{\bigwedge_{i,j \leq n} a_i \neq b_j\}$$

(for some $n \in \omega$) is inconsistent. I.e. we have

$$\models \psi(\bar{a}) \longrightarrow \left(\neg\varphi(\bar{b}) \ \vee \ \bigvee_{i,j \leq n} a_i = b_j\right).$$

Since the elements of the tuple $\bar{b}$ are not from C any expansion of $\mathfrak{A}_A$ to $\tau_{A \cup \{\bar{b}'\}}$ must have

$$\mathfrak{A}_{A,\bar{b}'} \models \neg\varphi(\bar{b}') \ \vee \ \bigvee_{i,j \leq n} a_i = b'_j.$$

Thus by definition of the universal quantifier,

$$\mathfrak{A}_A \models \forall y_1, \ldots, y_n \left(\neg\varphi(\bar{y}) \ \vee \ \bigvee_{i,j \leq n} a_i = y_j\right).$$

This implies that

$$\mathfrak{A} \models \exists x_1, \ldots, x_n \forall y_1, \ldots, y_n \left(\neg\varphi(\bar{y}) \ \vee \ \bigvee_{i,j \leq n} x_i = y_j\right)$$

But now $\exists\bar{x}\forall\bar{y}\left(\neg\varphi(\bar{y}) \ \vee \ \bigvee_{i,j\leq n} x_i = y_i\right)$ is a sentence in $\mathscr{L}(\tau_C)$. Since $\mathfrak{A}$ satisfies this sentence, it is an element in $\mathrm{Th}(\mathfrak{A}_C)$ which by assumption is equal to $\mathrm{Th}(\mathfrak{B}_C)$.

But, $\mathfrak{B}_B \models \varphi(\bar{b})$. Now we claim that for each $j \leq n$ the set

$$\{b' \mid \exists y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n \ \varphi(y_1, \ldots, y_{j-1}, b', y_{j+1}, \ldots, y_n)\}$$

is infinite: Since if not then, letting $\theta_j(\bar{x})$ be $\varphi(x_1, \ldots, x_{j-1}, b', x_{j+1}, \ldots, x_n)$, we would have that $\theta_j(\mathfrak{B})$ is finite. Now $b_j \in \theta_j(\mathfrak{B})$ and $\theta_j \in \mathscr{L}(\tau_C)$ so, *since C is algebraically closed*, $\theta_j(\mathfrak{B}) \subseteq C$ so $b_j \in C$. But this contradicts the assumption that all coordinates of $\bar{b}$ were not from C.

So with these infinitely many b''s we see that

$$\mathfrak{B}_C \models \forall\bar{x}\exists\bar{y} \ \varphi(\bar{y}) \ \wedge \ \bigwedge y_i \neq x_j.$$

But $\mathfrak{A}_C \equiv \mathfrak{B}_C$ and we already saw that $\mathfrak{A}_C$ does not satisfy the above sentence. Thus we have a contradiction, and so T must be consistent. □

We can also amalgamate many models at the same time.

Corollary. *If $\{\mathfrak{B}_i\}_{i\in I}$ is a nonempty set of τ-structures with $C \subseteq \mathfrak{B}_i$ a common subset, and $(\mathfrak{B}_i)_C \equiv (\mathfrak{B}_j)_C$ for all $i, j \in I$, then there exists a τ-structure $\mathfrak{D}$ which is an elementary extension of all the $\mathfrak{B}_i$'s. If furthermore C is algebraically closed, then as before we can arrange that $B_i \cap B_j = C$ (inside of D) for all $i \neq j$ from I.*

Proof. We do the proof without assuming C is algebraically closed. The modifications in the case where C is algebraically closed are much the same as before.

We use compactness together with induction. Let

$$\mathrm{T} := \bigcup_{i\in I} \mathrm{eldiag}(\mathfrak{B}_i).$$

We must check that T is satisfiable. By compactness T is consistent if and only if, for each finite $J \subseteq I$

$$\bigcup_{i\in J} \mathrm{eldiag}(\mathfrak{B}_i)$$

is consistent. This we can check by induction on $|J|$. If $|J| = 1$ then this is clear. For $|J| = n + 1$ let $J = J' \cup \{j\}$. By induction there is some $\mathfrak{D}_{J'}$ such that for all $j' \in J'$ we have $\mathfrak{B}_{j'} \preccurlyeq \mathfrak{D}_{J'}$. Now using the Elementary Amalgamation Theorem we can amalgamate $\mathfrak{D}_{J'}$ together with $\mathfrak{B}_j$ over C. □

We can use these results to realize as many types as we want.

Corollary. *Given any τ-structure $\mathfrak{C}$ there exists some elementary extension $\mathfrak{D}$ of $\mathfrak{C}$ such that for all $p \in S_1(C)$, p is realized in $\mathfrak{D}$.*

Proof. For $p \in S_1(C)$ we have seen that we can realize it in some extension, say $\mathfrak{B}_p$, where $\mathfrak{C} \preccurlyeq \mathfrak{B}_p$. Using the above corollary with the family $\{\mathfrak{B}_p : p \in S_1(C)\}$ we get the existence of $\mathfrak{D}$ such that for all $p \in S_1(C)$ we have $\mathfrak{C} \preccurlyeq \mathfrak{B}_p \preccurlyeq \mathfrak{D}$. Thus every $p \in S_1(C)$ is realized in $\mathfrak{D}$. $\square$

Remark. In the above corollary we could have taken some subset of $S_1(C)$ and realized all types from this subset. The proof is the same.

This will later be used to build *saturated* models, where every type over every "reasonably small" subset of the model, is realized in the model.