Math 225A – Model Theory

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Autumn 2013

General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory.*

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 2

Last time we introduced closed τ-terms. Before we start trying to make sense of how to interpret terms we must define the notions of expansions and reducts. Given two signatures τ and σ we write $\tau \subseteq \sigma$ when $\mathcal{C}_\tau \subseteq \mathcal{C}_\sigma$, $\mathcal{F}_\tau \subseteq \mathcal{F}_\sigma$ and $\mathcal{R}_\tau \subseteq \mathcal{R}_\sigma$.

Definition. Given signatures τ and σ with $\tau \subseteq \sigma$ and a σ-structure $\mathfrak{B}$ we may define a τ-structure $\mathfrak{A} = \mathfrak{B}|_\tau$. $\mathfrak{A}$ is the τ-structure given by $\mathrm{dom}(\mathfrak{A}) = \mathrm{dom}(\mathfrak{B})$ and for $x \in \mathcal{C}_\tau \cup \mathcal{F}_\tau \cup \mathcal{R}_\tau$ we set $x^{\mathfrak{A}} = x^{\mathfrak{B}}$. We call $\mathfrak{A}$ the **τ-reduct of $\mathfrak{B}$** and say that $\mathfrak{B}$ is *an* **expansion of $\mathfrak{A}$ to** σ.

This gives a functor from $\mathbf{Str}(\sigma)$ (the category of σ-structures) to $\mathbf{Str}(\tau)$[1].

As long as either $\mathrm{dom}(\mathfrak{A}) \neq \emptyset$ or $\mathcal{C}_\sigma = \emptyset$ then the τ-structure $\mathfrak{A}$ admits some expansion to σ.

Example. We may think of $\mathbb{R}$ as an ordered field $(\mathbb{R}, +, \cdot, \leq, 0, 1)$[2]. Then the signature of this structure is $\{+, \cdot, \leq, 0, 1\}$. Now we may form the reduct to, say, the language of groups $\{+, 0\}$. This yields the structure $(\mathbb{R}, +, 0)$. Of course there are many ways to expand the group structure on the reals.

Recall that for a signature τ, the set $\mathscr{T}(\tau)$ of all **closed τ-terms** is the smallest set of finite sequences from $\mathcal{C}_\tau \cup \mathcal{F}_\tau \cup \{(\} \cup \{,\} \cup \{)\}$, such that for all $c \in \mathcal{C}_\tau$ then $c \in \mathscr{T}(\tau)$ and such that if $t_1, \dots, t_n \in \mathscr{T}(\tau)$ and $f \in \mathcal{F}_\tau$ (with $\mathrm{arity}(f) = n$) then $f(t_1, \dots, t_n) \in \mathscr{T}(\tau)$.

Remark. To show that $\mathscr{T}(\tau)$ is actually a set one uses weak recursion: given a set X an element $a \in X$ and a function $I : X \to X$ then there is a unique function $f : \omega \to X$ such that $f(0) = a$ and for all $n \in \omega$ we have $f(n+1) = I(f(n))$. This is a theorem, which will be proven in the homework.

[1] morphisms of σ-structures respect of the σ-structure and so they will also respect all the τ-structure.

[2] This notation for a structure means that $\mathbb{R}$ is the domain, $+$ and $\cdot$ are the interpretations of the function symbols, $\leq$ is the interpretation of the relation symbol, and $0, 1$ are the interpretations of the constant symbols.

Definition. For τ a signature we define the **free term τ-structure** $\widetilde{\mathscr{T}(\tau)}$ to be the τ-structure with domain $\mathscr{T}(\tau)$ and with interpretations as follows:

- for $c \in \mathcal{C}_\tau$ we set $c^{\widetilde{\mathscr{T}(\tau)}} = c$
- for $f \in \mathcal{F}_\tau$ with arity$(f) = n$ and with $a_1, \ldots, a_n \in \mathscr{T}(\tau)$, we set
$$f^{\widetilde{\mathscr{T}(\tau)}}(a_1, \ldots, a_n) = f(a_1, \ldots, a_n).$$
- for $R \in \mathcal{R}_\tau$ we let $R^{\widetilde{\mathscr{T}(\tau)}} = \emptyset$.

It is clear that $\widetilde{\mathscr{T}(\tau)}$ is in fact a τ-structure. The real content of this fact is just the fact that $\mathscr{T}(\tau)$ actually forms a set.

Furthermore $\widetilde{\mathscr{T}(\tau)}$ has a universal property.

Proposition. *For any τ-structure $\mathfrak{A}$ then there exists a unique homomorphism of τ-structures $\rho : \widetilde{\mathscr{T}(\tau)} \to \mathfrak{A}$.*

Proof. We first define the map $\rho : \widetilde{\mathscr{T}(\tau)} \to \mathfrak{A}$ by recursion on the construction on $\mathscr{T}(\tau)$.

- for $c \in \mathcal{C}_\tau$ we let $\rho(c^{\widetilde{\mathscr{T}(\tau)}}) = c^{\mathfrak{A}}$.
- if $t \in \mathscr{T}(\tau)$ has the form $f(t_1, \ldots, t_n)$ then $\rho(t) = f^{\mathfrak{A}}(\rho(t_1), \ldots, \rho(t_n))$.

This is well-defined since we have a unique parsing lemma for terms [Hodges Problem 5, of Section 1.3]. Furthermore ρ is clearly a homomorphism. On constant and function symbols it is defined as is should be and for relation symbols the claim is vacuous since $R^{\widetilde{\mathscr{T}(\tau)}} = \emptyset$. This takes care of the existence.

For the uniqueness we use induction on the complexity of terms. Suppose $\rho, \xi : \widetilde{\mathscr{T}(\tau)} \to \mathfrak{A}$ are homomorphisms. Then

- for all $c \in \mathcal{C}_\tau$ we have $\rho(c) = c^{\mathfrak{A}} = \xi(c)$,
- if $t \in \mathscr{T}(\tau)$ has the form $f(t_1, \ldots, t_n)$ then $\rho(t) = f^{\mathfrak{A}}(\rho(t_1), \ldots, \rho(t_n)) = f^{\mathfrak{A}}(\xi(t_1), \ldots, \xi(t_n)) = \xi(t)$ since the t_i's have lower complexity than t.

Thus $\rho = \xi$ and this finishes the proof. □

The proposition shows that the free term structure, $\widetilde{\mathscr{T}(\tau)}$, is an initial object in $\mathbf{Str}(\tau)$. [As such it is the unique (up to isomorphism) τ-structure which satisfies the property of the proposition.].

This way of constructing structures out of their own names will be done several times during the course. If one wants to show that certain sentences are consistent or that it is possible to have some specific kind of structure, then one can try to write down what one wants to exist and then the description is *itself* the structure.

We now introduce one of the most important kinds of expansions.

Definition. Given a τ-structure $\mathfrak{M}$ and a subset $A \subseteq \operatorname{dom}(\mathfrak{M})$, then τ_A is the signature with

$$\mathcal{C}_{\tau_A} = \mathcal{C}_\tau \dot{\cup} A \qquad\qquad \text{(disjoint union)}$$

and with $\mathcal{F}_{\tau_A} = \mathcal{F}_\tau$, and $\mathcal{R}_{\tau_A} = \mathcal{R}_\tau$. We define $\mathfrak{M}_A$ to be the expansion of $\mathfrak{M}$ to τ_A by interpreting $a \in A \subseteq \mathcal{C}_{\tau_A}$ as $a^{\mathfrak{M}_A} = a$.

The expansion $\mathfrak{M}_A$ has names (in the form of constant symbols) for all the elements of A.

We sometimes want to talk about having *variables*. Variables should be thought of as constant symbols that we don't know how to interpret yet.

Definition. Given a signature τ, a **term** over τ is an element of $\mathscr{T}(\tau_X)$ where $X = \{x_i : i \in \omega\}$.

[strictly speaking we have not defined what τ_X means in the context where there is no τ-structure, i.e. no domain.]

Now let $\mathfrak{A}$ be a τ-structure. We shall interpret the terms in $\mathscr{L}(\tau_X)$ in $\mathfrak{A}$. Let t be a term in which only the variables x_i for $i < n$ occur, so that $t \in \mathscr{L}(\tau_{\{x_i : i<n\}})$. Then $t^{\mathfrak{A}} : A^n \to A$ is the function given by sending, for each $\bar{a} = (a_0, \ldots, a_{n-1}) \in A^n$ to the image of t under the unique $\tau_{\{x_i : i<n\}}$-homomorphism

$$\mathscr{T}(\widetilde{\tau_{\{x_i : i<n\}}}) \longrightarrow \mathfrak{A}_{\bar{a}}$$

where $x_i^{\mathfrak{A}_A} = a_i$.

Remark. If $A = \emptyset$ then there are no n-tuples $\bar{a} \in A^n$ and so we interpret t as the empty function.

Logic

Definition. Given a signature τ, an **atomic formula** is a finite sequence from the set

$$\mathcal{C}_\tau \cup \mathcal{F}_\tau \cup \mathcal{R}_\tau \cup \{(\} \cup \{,\} \cup \{)\} \cup \{=\}$$

of the form

- $t = s$, or
- $R(t_1, \ldots, t_n)$

where t, s and $t_1, \ldots, t_n$ are τ-terms, and $R \in \mathcal{R}_\tau$.

The set of all τ-formulae, $\mathscr{L}(\tau)$, is the smallest set of finite sequences in

$$\mathcal{C}_\tau \cup \mathcal{F}_\tau \cup \mathcal{R}_\tau \cup \{(\} \cup \{,\} \cup \{)\} \cup \{=\} \cup \{\vee\} \cup \{\wedge\} \cup \{\rightarrow\} \cup \{\leftrightarrow\} \cup \{\forall\} \cup \{\exists\} \cup \{x_i : i \in \omega\}$$

(where all unions are disjoint) such that

- every atomic formula belongs to $\mathscr{L}(\tau)$,
- if φ is in $\mathscr{L}(\tau)$, then $\neg(\varphi)$ is in $\mathscr{L}(\tau)$,
- if φ and ψ are in $\mathscr{L}(\tau)$, then $(\varphi \vee \psi), (\varphi \wedge \psi), (\varphi \to \psi)$ and $(\varphi \leftrightarrow \psi)$ are in $\mathscr{L}(\tau)$
- if φ is in $\mathscr{L}(\tau)$ and $i \in \omega$ then $(\exists x_i)\varphi$ and $(\forall x_i)\varphi$ are in $\mathscr{L}(\tau)$.

an element of $\mathscr{L}(\tau)$ is called a **formula**, and $\mathscr{L}(\tau)$ is called the **language** of τ.

Remark. Each of the four conditions on the set $\mathscr{L}(\tau)$ may be thought of as closure properties of $\mathscr{L}(\tau)$. For instance $\mathscr{L}(\tau)$ is closed under taking negation, i.e. if $\varphi \in \mathscr{L}(\tau)$ then $\neg(\varphi) \in \mathscr{L}(\tau)$. By the way we have defined $\mathscr{L}(\tau)$ it is clear that the set actually exists. This is since there is at least one set satisfying all the closure properties (namely the set of all sequences in the given symbols) and since each condition in the definition is such that for any collection of sets of sequences satisfying the given condition, their intersection will also satisfy it. Thus taking the intersection of all sets satisfying the conditions we get the smallest set, namely $\mathscr{L}(\tau)$.

This construction of $\mathscr{L}(\tau)$ is "from above". A more useful way to construct $\mathscr{L}(\tau)$ would be "from below" namely using weak recursion as in the above construction of the closed τ-terms.

Free and bound variables

We would like to say that a variable is free (or bound) in a formula φ but really we can only say that a particular instance of the given variable is free (or bound).

We define free and bound variables by recursion on the construction of formulas.

- In an atomic formula all variables are free, including variables not appearing in the atomic formula.
- In $\neg(\varphi)$, $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$ the free (respectively bound) instances of variables are what they where in the constituent formulas. For clarity let us be more precise in the case $(\varphi \wedge \psi)$. Now $(\varphi \wedge \psi)$ is a sequence of length, $3 + \text{length}(\varphi) + \text{length}(\psi)$. For $i < \text{length}(\varphi) + 1$ then the i^{th} coordinate is a free (respectively bound) variable if the $(i-1)^{th}$ coordinate of φ is free (respectively bound). For $2 + \text{length}(\varphi) \leq j < 3 + \text{length}(\varphi) + \text{length}(\psi)$ then the j^{th} coordinate is a free (respectively bound) variable if the $(j - (2 + \text{length}(\varphi)))^{th}$ coordinate is free (respectively bound) in ψ.
- In $(\forall x_i)\varphi$ and $(\exists x_i)\varphi$ no instance of x_i is free (i.e. all such instances are bound) and all other variables remain how they were (free or bound) in φ.

Warning!. The same variable can appear twice in the same formula as both a free and bound variable! For example in the formula $(\exists x_1(\neg(x_1 = x_2)) \wedge x_1 = x_3)$, the variable x_1 is bound in the first instance and free in the second. Of course it is not a good idea to do this, but it is allowed.

We can now understand how to interpret formulas. Given a τ-structure $\mathfrak{A}$ (with domain A) and φ a formula we interpret φ as follows

- If φ is atomic and equal to $t = s$ then $\varphi(\mathfrak{A}) := \{a \in \mathfrak{A}^\omega : t^{\mathfrak{A}}(a) = s^{\mathfrak{A}}(a)\}$.
- If φ is atomic and equal to $R(t_1, \dots, t_n)$ then $\varphi(\mathfrak{A}) := \{a \in A^\omega : (t_1^{\mathfrak{A}}(a), \dots, t_n^{\mathfrak{A}}(a)) \in R^{\mathfrak{A}}\}$.
- If φ is $\neg(\psi)$ then $\varphi(\mathfrak{A}) := A^\omega \setminus \varphi(\mathfrak{A})$.
- If φ is $(\psi \wedge \theta)$ then $\varphi(\mathfrak{A}) := \psi(\mathfrak{A}) \cap \theta(\mathfrak{A})$.
- If φ is $(\psi \vee \theta)$ then $\varphi(\mathfrak{A}) := \psi(\mathfrak{A}) \cup \theta(\mathfrak{A})$.
- If φ is $(\exists x_i)\psi$ then

  $\varphi(\mathfrak{A}) := \{a \in A^\omega : \exists b_i \in A \text{ such that } \tilde{a} \in \psi(\mathfrak{A}) \text{ where } (\tilde{a})_j = b_i \text{ if } j = i \text{ and } (\tilde{a})_j = a_j \text{ otherwise}\}$

- If φ is $(\forall x_i)\psi$ then

  $\varphi(\mathfrak{A}) := \{a \in A^\omega : \forall b_i \in A \text{ such that } \tilde{a} \in \psi(\mathfrak{A}) \text{ where } (\tilde{a})_j = b_i \text{ if } j = i \text{ and } (\tilde{a})_j = a_j \text{ otherwise}\}$

Remark. Often when $\varphi \in \mathscr{L}(\tau)$ is a formula and the free variables of φ are taken from $\{x_i : i < n\}$, we think of $\varphi(\mathfrak{A})$ as a subset of A^n. *This is a mistake.* By the above definition $\varphi(\mathfrak{A})$ is a subset of A^ω.

Definition. A formula $\varphi \in \mathscr{L}(\tau)$ is a **sentence** if no free variables appear in φ

Note that all variables not *appearing* in φ are free. So for a formula to be a sentence we only care about the variables actually appearing.

Definition. For φ a sentence we say that $\mathfrak{A}$ models φ, written $\mathfrak{A} \models \varphi$, if $\varphi(\mathfrak{A}) = A^\omega$.