Math 225A – Model Theory

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General Information

These notes are based on a course in *Metamathematics* taught by Professor Thomas Scanlon at UC Berkeley in the Autumn of 2013. The course will focus on Model Theory and the course book is Hodges' *a shorter model theory*.

As with any such notes, these may contain errors and typos. I take full responsibility for such occurences. If you find any errors or typos (no matter how trivial!) please let me know at mps@berkeley.edu.

Lecture 10

We have introduced the Ehrenfeucht-Fraïssé game $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$ as a way of testing for similarities between the structures $\mathfrak{A}$ and $\mathfrak{B}$. In particular we saw that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic then $\exists$ has a winning strategy (i.e. then $\mathfrak{A} \sim_\omega \mathfrak{B}$). The natural question is then: how similar are $\mathfrak{A}$ and $\mathfrak{B}$ if we know that $\exists$ has a winning strategy for $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$? The answer is somewhere between elementary equivalence and isomorphism.

Notation. Recall that $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ agree on all sentence of the infinitary language $\mathscr{L}_{\infty\omega}(\tau)$. In particular $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}$.

Theorem 1. *If τ is a countable signature, then $\mathfrak{A} \sim_\omega \mathfrak{B}$ if and only if $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$.*

Proof.
"$\Longrightarrow$" We will show by induction on the complexity of an $\mathscr{L}_{\infty\omega}(\tau)$-sentence Φ, for *any* signature τ, that $\mathfrak{A} \sim_\omega \mathfrak{B}$ implies $\mathfrak{A} \models \Phi \Leftrightarrow \mathfrak{B} \models \Phi$.

- If Φ is atomic then since $\mathfrak{A} \sim_\omega \mathfrak{B}$ implies $\mathfrak{A} \sim_0 \mathfrak{B}$ which implies that $\mathfrak{A} \models \Phi \Leftrightarrow \mathfrak{B} \models \Phi$.
- If Φ is $\neg\Psi$ then $\mathfrak{A} \models \Phi$ iff $\mathfrak{A} \not\models \Psi$ iff (by Induction Hypothesis) $\mathfrak{B} \not\models \Psi$ iff $\mathfrak{B} \models \Phi$.
- If Φ is $\bigvee \Xi$ then $\forall \xi \in \Xi$ we have $\mathfrak{A} \models \xi \Leftrightarrow \mathfrak{B} \models \xi$ which happens iff $\mathfrak{B} \models \Phi$. Likewise if Φ is $\bigwedge \Xi$.
- Suppose Φ is $\exists x \Psi(x)$ and that $\mathfrak{A} \models \Phi$. Then there exists $a \in A$ such that $\mathfrak{A}_a \models \Psi(a)$. By hypothesis $\exists$ has a winning strategy for $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$. Treating a as the 0th move of $\forall$ let $b \in B$ be the element that $\exists$ picks by way of her winning strategy. Then (a, b) is a winning position for $\exists$ in the game $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$. This is equivalent to $\exists$ having a winning strategy for the game $\mathrm{EF}_\omega(\mathfrak{A}_a, \mathfrak{B}_b)$ i.e. $\mathfrak{A}_a \sim_\omega \mathfrak{B}_b$. So by the induction hypothesis $\mathfrak{B}_b \models \Psi(b)$ i.e. $\mathfrak{B} \models \exists x \Psi(x)$. Thus $\mathfrak{B} \models \Phi$. Reversing the roles of $\mathfrak{A}$ and $\mathfrak{B}$ we see that $\mathfrak{B} \models \Phi$ also implies that $\mathfrak{A} \models \Phi$.

"$\Longleftarrow$": Conversely suppose $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. We claim that $\mathfrak{A} \sim_\omega \mathfrak{B}$. If $\forall$ plays $a \in A$ then $\exists$ will respond $b \in \mathfrak{B}$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$[1] in $\mathscr{L}_{\infty\omega}(\tau)$. More precisely let[2]

$$\Phi := \{\varphi(x) \in \mathscr{L}_{\infty\omega}(\tau) : \mathfrak{A} \models \varphi(a/x) \text{ with (number of symbols in } \varphi) \leq 2^{|\mathfrak{A}|+|\mathfrak{B}|+\aleph_0}\}.$$

Then $\mathfrak{A} \models \bigwedge \Phi(a)$ i.e. $\mathfrak{A} \models \exists x \bigwedge \Phi(x)$. By assumption $\mathfrak{B} \models \exists x \bigwedge \Phi(x)$. Let $b \in B$ be a witness. Then $\exists$ responds to $\forall$ by choosing the element b. Then $\mathfrak{A}_a \equiv_{\infty\omega} \mathfrak{B}_b$. Continuing in this way we get a ω-sequence which is a win for $\exists$. Following this procedure is thus a winning strategy for $\exists$, so $\mathfrak{A} \sim_\omega \mathfrak{B}$. $\square$

So $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$ does characterize an equivalence between $\mathfrak{A}$ and $\mathfrak{B}$ but in the very strong infinitary logic of $\mathscr{L}_{\infty\omega}$.

We will now slightly modify the game with the aim of getting a new game that exactly characterizes (for finite signatures) elementary equivalence, i.e. equivalence in first-order logic.

What follows relies heavily on the notion of unnested formulae. For convenience we repeat the definition.

Definition. An **unnested atomic formula** is one of the form

- $x = c$, for $c \in \mathcal{C}_\tau$ and x a variable.
- $F\bar{x} = y$ where $F \in \mathcal{F}_\tau$ and $\bar{x}, y$ are variables.
- $R\bar{x}$, where $R \in \mathcal{R}_\tau$ and $\bar{x}$ are variables.
- $x = y$, where x and y are variables.

An **unnested formula** is built from the unnested atomic formulae by the usual rules.

Definition. The **unnested Ehrenfeucht-Fraïssé game** $\mathrm{EF}_\alpha[\mathfrak{A}, \mathfrak{B}]$ (note the square brackets) is the game where at stage $\beta < \alpha$, $\forall$ chooses en element from either $\mathfrak{A}$ or $\mathfrak{B}$ (i.e. $a_\beta \in A$ or $b_\beta \in B$) and $\exists$ responds with an element from the other structure. $\exists$ wins if for every *unnested* atomic formula $\varphi(x_{\beta|_{\beta<\alpha}})$

$$\mathfrak{A} \models \varphi(\bar{a}) \qquad \Longleftrightarrow \qquad \mathfrak{B} \models \varphi(\bar{b}).$$

We write $\mathfrak{A} \approx_\alpha \mathfrak{B}$ when $\exists$ has a winning strategy in $\mathrm{EF}_\alpha[\mathfrak{A}, \mathfrak{B}]$.

Remark. Most often the ordinal α in the above definition will either be finite or will be ω.

[1] $\mathrm{tp}(a)$ is the *type* of x it is the set of all $\mathscr{L}_{\infty\omega}(\tau)$ sentences in one variable which are true of a.

[2] the reason for the somewhat odd bound on the number of symbols in the definition of Φ is to ensure that Φ is actually a set.

Remark. If $\mathfrak{A} \sim_\alpha \mathfrak{B}$ then $\mathfrak{A} \approx_\alpha \mathfrak{B}$. This is clear since a play in $\mathrm{EF}_\alpha[\mathfrak{A}, \mathfrak{B}]$ is in fact a play of the old game $\mathrm{EF}_\alpha(\mathfrak{A}, \mathfrak{B})$. So a winning strategy in $\mathrm{EF}_\alpha(\mathfrak{A}, \mathfrak{B})$ is also a winning strategy in $\mathrm{EF}_\alpha[\mathfrak{A}, \mathfrak{B}]$.

The converse is not true. For example take $\tau = \{0, 1\}$ where 0 and 1 are constant symbols. Let $\mathfrak{A}$ be a τ-structure where A has one element and where $0^{\mathfrak{A}} = 1^{\mathfrak{A}}$ and let $\mathfrak{B}$ be τ-structure with B having two elements where $0^{\mathfrak{B}} \neq 1^{\mathfrak{B}}$. Then $\mathfrak{A} \approx_0 \mathfrak{B}$ but $\mathfrak{A} \not\sim_0 \mathfrak{B}$. To see that $\mathfrak{A} \approx_0 \mathfrak{B}$ we must see that $\mathfrak{A}$ and $\mathfrak{B}$ agree on all *unnested* atomic sentences. But there are none! So they vacuously agree. In the other game however, the (nested) atomic sentence $0 = 1$ is satisfied by $\mathfrak{A}$ but not by $\mathfrak{B}$. Note however, that the unnested Ehrenfeucht-Fraïssé game *can* tell $\mathfrak{A}$ and $\mathfrak{B}$ apart at level 1, i.e. $\mathfrak{A} \not\approx_1 \mathfrak{B}$. To see this suppose $\forall$ picks $a \in A$ then $\exists$ must pick $b \in B$. But then thinking of the formula $x{=}0$ we see that $\mathfrak{A} \models a{=}0$ and $\mathfrak{B} \not\models b{=}0$. So $\exists$ cannot win $\mathrm{EF}_1[\mathfrak{A}, \mathfrak{B}]$.

Question. What is the relation between $\approx_\omega$ and $\sim_\omega$? I.e. do there exist $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \approx_\omega \mathfrak{B}$ and $\mathfrak{A} \not\sim_\omega \mathfrak{B}$?

Remark. There do exist $\mathfrak{A}$ and $\mathfrak{B}$ τ-structures (for finite signature τ) such that $\forall k < \omega \; \mathfrak{A} \approx_k \mathfrak{B}$ *but* $\mathfrak{A} \not\approx_\omega \mathfrak{B}$.

Here is an example. Let $\mathfrak{A} = (\mathbb{N}, <)$ and $\mathfrak{B} = (\mathbb{N} \oplus \mathbb{Z}, <)$ (where $\mathbb{N} \oplus \mathbb{Z}$ is the order gotten by adding a copy of $\mathbb{Z}$ after $\mathbb{N}$). Then $\mathfrak{A} \approx_k \mathfrak{B}$ for all $k < \omega$, *but* $\mathfrak{A} \not\approx_\omega \mathfrak{B}$. To see that $\mathfrak{A} \not\approx_\omega \mathfrak{B}$ imagine the case where $\forall$ picks all elements of $\mathbb{Z}$ (from $\mathfrak{B}$) doing down, then $\exists$ will run out of elements in $\mathbb{N}$ (from $\mathfrak{A}$) to pick.

Definition. For φ a formula, the **quantifier rank** $\mathrm{qr}(\varphi)$ is the number of nested quantifiers in φ. I.e.:

- If φ is atomic, then $\mathrm{qr}(\varphi) = 0$
- $\mathrm{qr}(\varphi \wedge \psi) = \mathrm{qr}(\varphi \vee \psi) = \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}$
- $\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi)$
- $\mathrm{qr}(\exists\varphi) = \mathrm{qr}(\varphi) + 1$.

Theorem 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures. Then $\mathfrak{A} \equiv \mathfrak{B}$ if and only if for all finite $\tau' \subseteq \tau$ we have $\mathfrak{A}|_{\tau'} \approx_k \mathfrak{B}|_{\tau'}$ for all $k < \omega$.*

Remark. Clearly $\mathfrak{A} \equiv \mathfrak{B}$ iff for all finite $\tau' \subseteq \tau$ $\mathfrak{A}|_{\tau'} \equiv \mathfrak{B}|_{\tau'}$. Thus it suffices to prove the theorem in the case where τ is finite. The statement then becomes that $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{A} \approx_k \mathfrak{B}$ for all $k < \omega$.

Before giving a proof of this theorem we need an important lemma. Hodges calls it the *Fraïssé-Hintikka theorem* and notes that it is "the fundamental theorem about the equivalence relations $\approx_k$". The theorem will follow as a corollary of the lemma.

Lemma. *For a finite signature τ and $k, n < \omega$, there is a finite set $\Theta_{n,k}$ of unnested formulae of quantifier rank $\leq k$ in n free variables $x_0, \ldots, x_{n-1}$, such that*

0. Distinct elements of $\Theta_{n,k}$ are inconsistent, i.e. for any $\eta, \theta \in \Theta_{n,k}$ then

$$\models \forall \bar{x}(\eta \to \neg\theta) \ .^3$$

1. If $\varphi \in \mathscr{L}(\tau)$ has quantifier rank $\leq k$ and free variables $x_0, \ldots, x_{n-1}$ then there is some subset $\Phi \subseteq \Theta_{n,k}$ such that $\models \forall \bar{x}(\varphi \leftrightarrow \bigvee \Phi)$.

2. Given $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ then for any n-tuples $\bar{a} \in A^n$ and $\bar{b} \in B^n$, we have $\mathfrak{A}_{\bar{a}} \approx_k \mathfrak{B}_{\bar{b}}$ if and only if for each $\theta \in \Theta_{n,k}$,

$$\mathfrak{A} \models \theta(\bar{a}) \qquad \Longleftrightarrow \qquad \mathfrak{B} \models \theta(\bar{b}).$$

Notation. For φ a formula define $\varphi^{[0]} := \varphi$ and $\varphi^{[1]} := \neg\varphi$.

Proof. We first construct $\Theta_{n,k}$ by recursion on k. [Note: this does not mean that we fix n. In the induction step we will use the $n+1$ level]

Let Φ be the set of unnested atomic formulae in $\mathscr{L}(\tau)$ in variables $x_0, \ldots, x_{n-1}$. This set is finite. This is because τ is finite and to construct an unnested atomic formulae we are only allowed to introduce one symbol from τ.

To get $\Theta_{n,0}$ we will go through every way one might choose to make each instance of $\varphi \in \Phi$ either true or false, and then take conjunctions of these formulae. More precisely we let

$$\Theta_{n,0} := \left\{ \bigwedge_{\varphi \in \Phi} \varphi^{[s(\varphi)]} \mid s : \Phi \longrightarrow \{0,1\} \right\}$$

and then

$$\Theta_{n,k+1} := \left\{ \bigwedge_{\varphi \in Y} \exists x_n \ \varphi(\bar{x}, x_n) \wedge \bigwedge_{\psi \in Z} \forall x_n \ \neg\psi(\bar{x}, x_n) \ \Big| \ \text{for } Y, Z \text{ a partition of } \Theta_{n+1,k} \right\}.$$

This finishes the construction of the sets $\Theta_{n,k}$ for arbitrary $n, k < \omega$.

Now we must check that conditions *0)*, *1)* , and *2)* are satisfied. First note that $\Theta_{n,k}$ is indeed finite and all elements are unnested and have quantifier rank $\leq k$.

0. Condition *0)* is reasonably clear. For $k = 0$ and $s, t : \Phi \longrightarrow \{0,1\}$ with $s \neq t$ there is some φ such that $s(\varphi) \neq t(\varphi)$ then

$$\left(\bigwedge_{\psi \in \Phi} \psi^{[s(\psi)]} \right) \longrightarrow \varphi^{[s(\varphi)]}$$

[3] The notation $\models \psi$ just means that for every τ-structure $\mathfrak{A}$ we have $\mathfrak{A} \models \psi$.

and

$$\left(\bigwedge_{\psi\in\Phi}\psi^{[t(\psi)]}\right)\longrightarrow\varphi^{[t(\varphi)]}.$$

Since $\varphi^{[s(\varphi)]}$ and $\varphi^{[t(\varphi)]}$ are explicitly inconsistent we see that $\bigwedge\psi^{[s(\psi)]}$ and $\bigwedge\psi^{[t(\psi)]}$ are inconsistent as well.
For the level $k+1$, suppose $Y\neq Y'$ and let $\eta\in Y\setminus Y'$. Now consider two formulas from $\Theta_{n,k+1}$. Then

$$\left(\bigwedge_{\varphi\in Y}\exists x_n\ \varphi\wedge\bigwedge_{\psi\in Y^c}\forall x_n\ \neg\psi\right)\longrightarrow\exists x_n\ \eta$$

whereas

$$\left(\bigwedge_{\varphi\in Y'}\exists x_n\ \varphi\wedge\bigwedge_{\psi\in(Y')^c}\forall x_n\ \neg\psi\right)\longrightarrow\forall x_n\neg\eta.$$

and since $\forall x_n\neg\eta\leftrightarrow\neg\exists x_n\ \eta$ we have an explicit inconsistency. By induction condition *0)* holds for all the sets $\Theta_{n,k}$.

1. To see that *1)* holds, note that if φ is an unnested formula of quantifier rank 0 in n free variables, then φ is a boolean combination of elements of Φ and so equivalent to some element of $\Theta_{n,0}$.
   [Case $\mathrm{qr}(\varphi)\leq k+1$?????]
2. We show that condition *2)* holds by induction on k.
   - For $k=0$. $(\mathfrak{A},\bar{a})\approx_0(\mathfrak{B},\bar{b})$ means that for ψ an unnested atomic τ-formula, $\mathfrak{A}\models\psi(\bar{a})\iff\mathfrak{B}\models\psi(\bar{b})$. But the formulae in $\Theta_{n,0}$ are exactly the atoms in the boolean algebra generated by unnested atomic formulae. So if $(\mathfrak{A},\bar{a})$ and $(\mathfrak{B},\bar{b})$ agree on the unnested atomic formulae then they will agree on all elements of $\Theta_{n,0}$, and vice versa.
   - At stage $k+1$ we will take one implication at a time. First suppose $(\mathfrak{A},\bar{a})\approx_{k+1}(\mathfrak{B},\bar{b})$. We show that for all $\varphi\in\Theta_{n,k+1}$, $\mathfrak{A}\models\varphi(\bar{a})$ implies $\mathfrak{B}\models\varphi(\bar{b})$. By symmetry we will also get that $\mathfrak{B}\models\varphi(\bar{b})$ implies $\mathfrak{A}\models\varphi(\bar{a})$. Let $\varphi\in\Theta_{n,k+1}$. By construction φ is

     $$\bigwedge_{\eta\in Y}\exists x_n\ \eta\wedge\bigwedge_{\xi\in Y^c}\forall x_n\ \neg\xi$$

     for some subset $Y\subseteq\Theta_{n+1,k}$. Suppose $\mathfrak{A}\models\varphi(\bar{a})$. For $\eta\in Y$ this implies that $\mathfrak{A}\models\exists x_n\eta(\bar{a},x_n)$. Let $c\in A$ be a witness to this, i.e. $\mathfrak{A}\models\eta(\bar{a},c)$. By hypothesis there exists some $d\in\mathfrak{B}$ such that $(\mathfrak{A},\bar{a},c)\approx_k(\mathfrak{B},\bar{b},d)$. Then by the induction hypothesis $\mathfrak{B}\models\eta(\bar{b},d)$ so $\mathfrak{B}\models\exists x_n\eta(\bar{b},x_n)$. So for each $\eta\in Y$ we have $\mathfrak{B}\models\exists x_n\eta(\bar{b},x_n)$. Likewise for $\xi\in Y^c$, if $\mathfrak{B}\not\models$

$\forall x_n \neg\xi(\bar{b}, x_n)$ then $\mathfrak{B} \models \exists x_n \xi(\bar{b}, x_n)$ and by same argument we have that $\mathfrak{A} \models \exists x_n \xi(\bar{a}, x_n)$. Since this is not true by assumption we must have $\mathfrak{B} \models \forall x_n \neg\xi(\bar{b}, x_n)$. Thus $\mathfrak{B} \models \varphi(b)$. By symmetry of the roles of $\mathfrak{A}$ and $\mathfrak{B}$ we have that $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{B} \models \varphi(\bar{b})$ for all $\varphi \in \Theta_{n,k+1}$.
Now for the converse implication. Suppose $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ agree on all of the $\Theta_{n,k+1}$ formulae. We must show $\mathfrak{A} \approx_{k+1} \mathfrak{B}$, i.e. that $\exists$ has a winning strategy in $\mathrm{EF}_{k+1}[\mathfrak{A}, \mathfrak{B}]$. Suppose $\forall$ plays $c \in A$. As $\Theta_{n+1,k}$ partitions A^{n+1} (by property *1)* of this lemma), there is exactly one formula $\eta \in \Theta_{n+1,k}$ such that $\mathfrak{A} \models \eta(\bar{a}, c)$. Now as $\Theta_{n,k+1}$ partitions A^n there is exactly one formula $\varphi \in \Theta_{n,k+1}$ such that $\mathfrak{A} \models \varphi(\bar{a})$. Then

$$\varphi(\bar{x}) \longrightarrow \exists x_n \eta(\bar{x}, x_n)$$

since φ either implies $\exists x_n \eta(\bar{x}, x_n)$ or $\forall x_n \neg\eta(\bar{x}_n, n)$, but we know that $\mathfrak{A} \models \eta(\bar{a}, c)$. By hypothesis $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ agree on formulae from $\Theta_{n,k+1}$ so $\mathfrak{B} \models \varphi(\bar{b})$. This in turn implies that $\mathfrak{B} \models \exists x_n \eta(\bar{b}, x_n)$. Let $d \in B$ be a witness. Then $\exists$ will play d. By the induction hypothesis $(\mathfrak{A}, \bar{a}, c) \approx_k (\mathfrak{B}, \bar{b}, d)$. Likewise if $\forall$ picks some $d \in B$ then $\exists$ can find $c \in A$ such that $(\mathfrak{A}, \bar{a}, c) \approx_k (\mathfrak{B}, \bar{b}, d)$. Thus $(\mathfrak{A}, \bar{a}) \approx_{k+1} (\mathfrak{B}, \bar{b})$.
By induction we now have the desired equivalence.

□

We can now prove the theorem as a corollary. For convenience we state the result again.

Theorem 3. *For τ finite and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ the following are equivalent.*

- $\mathfrak{A} \equiv \mathfrak{B}$
- $\mathfrak{A} \approx_k \mathfrak{B}$ *for all $k < \omega$.*

Proof. Suppose first that $\mathfrak{A} \equiv \mathfrak{B}$. We show by induction on k that $\mathfrak{A} \approx_k \mathfrak{B}$ for all k. For $k = 0$ we have $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \sim_0 \mathfrak{B}$, in particular $\mathfrak{A} \approx_0 \mathfrak{B}$.

Now for $k + 1$. Suppose $\forall$ picks $b \in B$. Let $\varphi \in \Theta_{1,k}$ be the unique element of $\Theta_{1,k}$ such that $\mathfrak{B} \models \varphi(b)$. Then $\mathfrak{B} \models \exists x_0 \varphi(x_0)$. This is a sentence, and so by assumption $\mathfrak{A} \models \exists x_0 \varphi(x_0)$. Let $a \in A$ be a witness. Thus $\mathfrak{A} \models \varphi(a)$. So $(\mathfrak{A}, a) \models \psi$ if and only if $(\mathfrak{B}, b) \models \psi$ for all $\psi \in \Theta_{1,k+1}$ (since both $\mathfrak{A}$ and $\mathfrak{B}$ don't satisfy any other of the elements of $\Theta_{1,k+1}$ apart from φ). Now by property *2)* of the lemma we have that $(\mathfrak{A}, a) \approx_k (\mathfrak{B}, b)$. Now since b was arbitrary (and the roles for $\mathfrak{A}$ and $\mathfrak{B}$ were unimportant) we have $\mathfrak{A} \approx_{k+1} \mathfrak{B}$. By induction we are done.

Conversely, suppose $\mathfrak{A} \approx_k \mathfrak{B}$ for all $k < \omega$. We must show that $\mathfrak{A} \equiv \mathfrak{B}$. We show by induction on r that if $\varphi \in \mathscr{L}(\tau)$ is unnested and $\mathrm{qr}(\varphi) \leq r$ then $\mathfrak{A}$ and $\mathfrak{B}$

agree on φ. Since we have already seen that all formulae are equivalent to unnested formulae this will finish the proof.

For $r = 0$, φ is an unnested atomic formula. Then since $\mathfrak{A} \approx_0 \mathfrak{B}$, $\mathfrak{A}$ and $\mathfrak{B}$ must agree on φ. Similarly for φ a boolean combination of unnested atomic formulae.

For $r + 1$, suppose φ is $\exists x \theta(x)$ with $\mathrm{qr}(\theta) \leq r$. Suppose $\mathfrak{A} \models \varphi$ and let $a \in A$ be a witness, i.e. $\mathfrak{A} \models \theta(a)$. Let $\psi \in \Theta_{1,r}$ be such that $\mathfrak{A} \models \psi(a)$. ψ is unique by *1)* above. Since $\mathfrak{A} \approx_{r+1} \mathfrak{B}$ there exists $b \in B$ such $(\mathfrak{A}, a) \approx_k (\mathfrak{B}, b)$, i.e. $\mathfrak{B} \models \psi(b)$. But since $\mathrm{qr}(\theta) \leq r$ we have by property *1)* of the lemma, that $\theta \leftrightarrow \bigvee_{\eta \in Y} \eta$ for some $Y \subseteq \Theta_{1,r}$. Thus $\psi(b) \longrightarrow \theta(b)$. So $\mathfrak{B} \models \exists x \theta(x)$, i.e. $\mathfrak{B} \models \varphi$. This completes the proof. □