Harish-Chandra bimodules in complex rank

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Structure of the talk

References

I will cover results from two of my preprints arXiv:2002.01555 and arXiv:2107.03173 .

I. Deligne categories

- Definition and properties.
- Ultraproduct construction.

II. Harish-Chandra bimodules

- Classical case.
- Central characters of Harish-Chandra bimodules in Deligne categories.
- Harish-Chandra bimodules of finite K-type.

Deligne categories

Intuitive definition

The categories $\mathrm{Rep}(GL_t)$ are interpolations of the categories of representations of groups GL_n to complex values of n .

Definition

$\mathrm{Rep}(GL_t)$ is the Karoubi envelope of the symmetric rigid monoidal category generated by a single object V of dimension t, such that $\mathrm{End}(V^{\otimes k}) \simeq \mathbb{C}[S_k]$ and $\mathrm{Hom}(V^{\otimes k}, V^{\otimes l}) = 0$ unless $k = l$.

- Karoubi envelope $\rightsquigarrow$ Formally adjoin images of all idempotents and finite direct sums.
- V has dim t $\rightsquigarrow$ $ev \circ \tau \circ coev : \mathbb{1} \to V \otimes V^* \to V^* \otimes V \to \mathbb{1}$ is the multiplication by t.
- Every indecomposable object is a direct summand in $[r, s] := V^{\otimes r} \otimes (V^*)^{\otimes s}$.

Deligne categories

Remark

I will limit myself to considering only the type A case, that is the categories $\mathrm{Rep}(GL_t)$.
All the definitions and results can be also generalized for $\mathrm{Rep}(O_t)$ and $\mathrm{Rep}(Sp_{2t})$ (was done jointly with Serina Hu).

- $\mathrm{Hom}([r,s],[r',s']) = \mathrm{Hom}(V^{\otimes r+s'}, V^{\otimes r'+s}) = 0$ unless $r+s' = r'+s$.
- $\mathrm{End}([r,s]) \simeq B_{r,s}(t)$ - walled Brauer algebra .

There is an injective map $i : \mathbb{C}[S_r \times S_s] \to B_{r,s}(t)$ (because $\mathbb{C}[S_k] \simeq \mathrm{End}(V^{\otimes k}) \simeq \mathrm{End}((V^*)^{\otimes k})$).

It has a splitting $\pi : B_{r,s}(t) \to \mathbb{C}[S_r \times S_s]$.

Indecomposable objects

We need to classify primitive idempotents in $B_{r,s}(t)$.

Theorem (Comes, Wilson)

For a partition $\nu \vdash k$ let z_ν denote the corresponding primitive idempotent in $\mathbb{C}[S_k]$.
For any bipartition (λ, μ) with $\lambda \vdash r, \mu \vdash s$ there exists a unique primitive idempotent $e_{\lambda,\mu} \in B_{r,s}(t)$ such that $\pi(e_{\lambda,\mu}) = z_\lambda \otimes z_\mu$.
And this gives the full classification of primitive idempotents in $B_{r,s}(t)$.

Corollary

Indecomposable objects in $\mathrm{Rep}(GL_t)$ are labeled by bipartitions $(\lambda, \mu) \mapsto V_{\lambda,\mu}$.
If $\lambda \vdash r, \mu \vdash s$, then $V_{\lambda,\mu}$ is a direct summand of $[r, s]$.

Properties of Deligne categories

- **The universal property.**
  Let $\mathcal{D}$ be a symmetric tensor category. Then
  $$\{F : \operatorname{Rep}(GL_t) \to \mathcal{D}\} \leftrightarrow \{X \in \mathcal{D} \text{ of dimension } t\}$$
  via $F \mapsto F(V) = X$.
- **Corollary.** If $t = n \in \mathbb{Z}_{>0}$ we have a symmetric tensor functor $F : \operatorname{Rep}(GL_t) \to \operatorname{Rep} GL(n, \mathbb{C})$, s.t. $F(V) = V^{(n)}$ (the tautological n-dimensional representation).
  Suppose $\operatorname{len}(\lambda) = l, \ \operatorname{len}(\mu) = m$ and $n \geq l + m$. Define
  $$[\lambda, \mu]_n := (\lambda_1, \ldots, \lambda_l, 0, \ldots, 0, -\mu_m, \ldots, -\mu_1) \in \Lambda^+(\mathfrak{gl}_n).$$
  Then for $t = n \geq l + m$ we have $F(V_{\lambda,\mu}) = V^{(n)}_{[\lambda,\mu]_n}$ (and if $t = n < l + m$ then $F(V_{\lambda,\mu}) = 0$).
- $\operatorname{Rep}(GL_t)$ is abelian (and semisimple) if and only if $t \notin \mathbb{Z}$.

Ultraproduct construction

Definition

An ultrafilter $\mathcal{F}$ on $S \neq \emptyset$ is a set of subsets of S, satisfying the following properties:

- For any $U_1, U_2 \in \mathcal{F}$ we have $U_1 \cap U_2 \in \mathcal{F}$,
- For any $U \subset S$ exactly one of $U, S - U$ is in $\mathcal{F}$,
- If $U_1 \subset U_2$ and $U_1 \in \mathcal{F}$ then $U_2 \in \mathcal{F}$.

Remark. Ultrafilters are the same as $\mathbb{F}_2$-valued characters of the ring of $\mathbb{F}_2$-valued functions on S.

Example

Fix some $s \in S$. A **principal** ultrafilter $\mathcal{F}_s$ consists of all subsets of S containing s.

It can be shown that there exists a non-principal ultrafilter on $\mathbb{N}$ and that such ultrafilter must contain all cofinite sets.

Ultraproduct construction

Let us fix a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.

Definition

For a collection of nonempty sets $X_n, \ n \in \mathbb{N}$ we can define their ultraproduct as follows

$$\prod_{\mathcal{F}} X_n = \prod X_n / \sim,$$

where we say $(x_1, x_2, x_3, \ldots) \sim (x'_1, x'_2, x'_3 \ldots)$ if $x_k = x'_k$ for almost all k, that is for all k in some $U \in \mathcal{F}$.

Łoś's theorem, roughly

Any first order language statement that is true for almost all X_n is also true for $\prod_{\mathcal{F}} X_n$.

Ultraproduct construction

Example

If all X_n are groups/algebras/fields then so is $\prod_{\mathcal{F}} X_n$.
If X_n are vector spaces over $\mathbb{F}_n$ then $\prod_{\mathcal{F}} X_n$ is a vector space over $\prod_{\mathcal{F}} \mathbb{F}_n$.

Non-example

If V_n are finite-dimensional vector spaces then $\prod_{\mathcal{F}} V_n$ is not necessarily finite-dimensional. However, if the dimensions of V_n are universally bounded, it will be.

Example

Take $\mathbb{F}_n = \overline{\mathbb{Q}}$. Then $\prod_{\mathcal{F}} \mathbb{F}_n$ is an algebraically closed field of characteristic zero and cardinality continuum. Therefore, by Steinitz's theorem it is (non-canonically) isomorphic to $\mathbb{C}$.

Ultraproduct construction

Definition

For a collection of small categories $\mathcal{C}_n,\ n \in \mathbb{N}$ we can define the category $\mathcal{C} = \prod_{\mathcal{F}} \mathcal{C}_{\mathbf{n}}$ via $Ob\ \mathcal{C} = \prod_{\mathcal{F}} Ob\ \mathcal{C}_n$ and for any pair of objects $X = \prod_{\mathcal{F}} X_n, Y = \prod_{\mathcal{F}} Y_n$ we define $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \prod_{\mathcal{F}} \mathrm{Hom}_{\mathcal{C}_n}(X_n, Y_n)$.

Theorem (Deligne)

Let $t \in \mathbb{C}$ be transcendental. The category $\mathrm{Rep}(GL_t)$ is isomorphic to the Karoubi envelope of the symmetric rigid monoidal subcategory in $\prod_{\mathcal{F}} \mathrm{Rep}\ \ GL(n, \overline{\mathbb{Q}})$ generated by the object $V = \prod_{\mathcal{F}} V^{(n)}$, where $V^{(n)}$ is the tautological n-dimensional representation of $GL(n, \overline{\mathbb{Q}})$. The $\mathbb{C}$-linear structure on $\mathrm{Rep}(GL_t)$ comes from choosing an isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$ with $(1, 2, 3, \ldots) \mapsto t \in \mathbb{C}$.

I will assume that t is transcendental from now on.

Lie algebra $\mathfrak{gl}_t$

We can define the Lie algebra $\mathfrak{gl}_t = V \otimes V^* = \prod_{\mathcal{F}} \mathfrak{gl}_n$ in $\mathrm{Rep}(GL_t)$. There is a **natural** action of $\mathfrak{gl}_t$ on every object of $\mathrm{Ind}\,\mathrm{Rep}(GL_t)$.

We can define $U = U(\mathfrak{gl}_t)$ as the quotient of the tensor algebra $T(\mathfrak{gl}_t)$ by the standard commutator relations. It has a PBW filtration $F^k U$ coming from the filtration of $T(\mathfrak{gl}_t)$. We have $F^k U = \prod_{\mathcal{F}} F^k U(\mathfrak{gl}_n)$.

Moreover, if $\mathcal{Z}$ is the center of U (i.e. $\mathcal{Z} = \mathrm{Hom}(\mathbb{1}, U)$), we have $F^k \mathcal{Z} = \prod_{\mathcal{F}} F^k \mathcal{Z}(U(\mathfrak{gl}_n))$. Therefore, $\mathcal{Z} = \mathbb{C}[z_1, z_2, z_3, \ldots]$ with $\deg z_k = k$.

Let me specify a particular choice of generators in $\mathcal{Z}$. We take $z_i = \prod_{\mathcal{F}} z_i^{(n)}$, where $z_i^{(n)}$ acts on M_χ via $\sum_{l=1}^{n} \chi_l^i$.

Given a central character $\theta : \mathcal{Z} \to \mathbb{C}$, define the **exponential generating function**: $\theta(u) = 1 + \sum_{i \geq 1} \frac{1}{i!} \theta(z_i) u^i$.

U-bimodules

Let $\mathcal{C}$ be either $\mathrm{Rep}(GL_t)$ or $\mathrm{Rep}_{\mathbb{C}}\, G$ for some reductive group G.

Let $\mathfrak{g} \in \mathcal{C}$ be the corresponding Lie algebra object. It acts **naturally** on all objects of $\mathrm{Ind}\,\mathcal{C}$.

Remark

We can define this for any symmetric tensor category $\mathcal{C}$.

Notations

Let $\mathfrak{g}^{op}$ be the opposite Lie algebra.
Let $U^2 := U(\mathfrak{g}) \otimes U(\mathfrak{g}^{op})$,
$\mathcal{Z}^2 = \mathcal{Z} \otimes \mathcal{Z} = \mathcal{Z}(U^2)$.
Let $\mathfrak{k} \simeq \mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}^{op}$ be the (anti)diagonal subalgebra.
Then we can consider any U^2-module $Y \in \mathrm{Ind}\,\mathcal{C}$ as a U-bimodule and $Y|_{\mathfrak{k}} = Y^{\mathrm{ad}}$.

$\mathfrak{k}$-algebraic bimodules

Definition

We say that a U-bimodule Y in $\operatorname{Ind}\mathcal{C}$ is **$\mathfrak{k}$-algebraic** if the action of $\mathfrak{k}$ on Y coincides with the **natural** action.

Lemma

$\{\mathfrak{k}$-*algebraic bimodules*$\} \leftrightarrow \{$*left* U*-modules in* $\operatorname{Ind}\mathcal{C}\}$.

Definition

U-bimodule $Y \in \operatorname{Ind}\mathcal{C}$ is called **finitely-generated** if it is a quotient of $U \otimes X \otimes U$ for some $X \in \mathcal{C}$. We say Y is generated by X.

Example

For any $X \in \mathcal{C}$ let $\Phi_X = X \otimes U$.
Then $\Phi_X^{\mathrm{ad}} = X \otimes U^{\mathrm{ad}} = X \otimes S(\mathfrak{g})$, so Φ_X is $\mathfrak{k}$-algebraic. And it is finitely generated as it is a quotient of $U \otimes X \otimes U$.

Harish-Chandra bimodules

Question

Now, how do we define a Harish-Chandra bimodule in $\mathcal{C}$?

There are two natural ways to do this. Let $Y \in \operatorname{Ind}\mathcal{C}$ be a finitely generated $\mathfrak{k}$-algebraic bimodule.

1. We can ask that $[Y^{\mathrm{ad}} : X] < \infty$ for any $X \in \mathcal{C}$, or
2. We can ask that $\mathcal{Z}^2$ acts finitely on Y (that is $\operatorname{Ann}_{\mathcal{Z}^2}(Y)$ is an ideal of finite codimension).

Lemma

Condition 1 implies condition 2.

Proof.

$\mathcal{Z}^2$ acts on $[Y : X] := \operatorname{Hom}_{\operatorname{Ind}\mathcal{C}}(X, Y^{\mathrm{ad}})$. Since it is finite dimensional, $\dim(\mathcal{Z}^2/\operatorname{Ann}_{\mathcal{Z}^2}[Y : X]) < \infty$. Now, if X generates Y, $\operatorname{Ann}_{\mathcal{Z}^2}(Y) = \operatorname{Ann}_{\mathcal{Z}^2}[Y : X]$. □

Difference between classical and complex rank cases

Corollary of Kostant's theorem

If $\mathcal{C} = \mathrm{Rep}_{\mathbb{C}} G$ then condition **2** implies condition **1**.

Definition

We say that Y is a **Harish-Chandra bimodule** if it satisfies condition **2**. Denote by $\mathcal{HC}$ the corresponding category.

Definition

If Y satisfies condition **1**, we say that it is a Harish-Chandra bimodule **of finite K-type**.

Example

Let $\theta : \mathcal{Z} \to \mathbb{C}$ be some central character of $U(\mathfrak{gl}_t)$ and let $U_\theta := U/\operatorname{Ker}(\theta)U$. Then $U_\theta \in \mathcal{HC}$, but is not of finite K-type.

Central characters

Because of condition **2**, we have a block decomposition for $\mathcal{HC}$:

$$\mathcal{HC} = \bigoplus \widetilde{\mathcal{HC}}(\theta_1, \theta_2),$$

where $\widetilde{\mathcal{HC}}(\theta_1, \theta_2)$ is the subcategory on which $\mathcal{Z}^2 = \mathcal{Z} \otimes \mathcal{Z}$ acts with generalized central character $\theta_1 \otimes \theta_2$.

Question

For which pairs (θ_1, θ_2) is the category $\widetilde{\mathcal{HC}}(\theta_1, \theta_2)$ non-zero?

Remark

Let $\mathcal{HC}(\theta_1, \theta_2)$ be the subcategory on which $\mathcal{Z}^2$ acts via $\theta_1 \otimes \theta_2$. It is enough to answer the questions for these categories.

Lemma

Any object Y in $\mathcal{HC}(\theta_1, \theta_2)$ is a quotient of $\Phi_X(\theta_2) = X \otimes U_{\theta_2}$.

Classification of central characters. Classical case

The answer in the classical case

Let $\mathcal{C} = \mathrm{Rep}_{\mathbb{C}}\, G$. Then $\mathcal{HC}(\theta_1, \theta_2)$ is nonzero if and only if there exist Verma modules M_{χ_1}, M_{χ_2}, such that $\mathcal{Z}$ acts via θ_i on M_{χ_i} and $\chi_1 - \chi_2 \in \Lambda(G)$.

Idea of proof

Any module is a quotient of $\Phi_X(\theta_2)$. If $\mathcal{Z}$ acts via θ_2 on M_χ then

$$\Phi_X(\theta_2) = X \otimes U_{\theta_2} \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(M_\chi, X \otimes M_\chi),$$

and $X \otimes M_\chi$ has filtration by $M_{\chi+\lambda}$, where $\lambda \in \Lambda(G)$ are weights of X.

Classification of central characters. Complex rank

Let $\mathcal{C} = \mathrm{Rep}(GL_t)$.

Main Theorem

The category $\mathcal{HC}(\theta_1, \theta_2)$ is non-zero if and only if

$$\theta_1(u) - \theta_2(u) = \sum_{i=1}^{r} (e^{(b_i+1)u} - e^{b_i u}) - \sum_{j=1}^{s} (e^{(c_j+1)u} - e^{c_j u})$$

for some $r, s \geq 0$ and $b_i, c_j \in \mathbb{C}$.

Remark. For O_t and Sp_{2t} we have that $\mathcal{HC}(\theta_1, \theta_2)$ is non-zero if and only if for some $r \geq 0$ and $b_i \in \mathbb{C}$:

$$\theta_1(u) - \theta_2(u) = \sum_{i=1}^{r} (\cosh((b_i + 1)u) - \cosh(b_i u))$$

Idea of proof

1. Any bimodule $Y \in \mathcal{HC}(\theta_1, \theta_2)$ is a quotient of $\Phi_X(\theta_2) = X \otimes U_{\theta_2}$. And any X is a quotient of $[r, s]$.
2. We want to understand for which θ_1 the quotient

$$\Phi_X(\theta_1, \theta_2) = (X \otimes U_{\theta_2})_{\theta_1} := X \otimes U_{\theta_2}/(z - \theta_1(z))(X \otimes U_{\theta_2})$$

   is non-zero (enough to take $X = [r, s]$).
3. Do this by induction on $r + s$ in the classical setting. Then take the ultraproduct.

Notation. Let $\eta : \mathfrak{h}^* \to \{\mathcal{Z}(\mathfrak{gl}_n) \to \bar{\mathbb{Q}}\}$ be the map that sends a weight to the corresponding central character.

Idea of proof

Example

Let us do the case when $[r, s] = [1, 0]$.

In the classical setting we have
$\Phi_{V^{(n)}}(\theta_2) = V^{(n)} \otimes U_{\theta_2} \subset \mathrm{Hom}_{\overline{\mathbb{Q}}}(M_\chi, V^{(n)} \otimes M_\chi)$, where $\eta(\chi) = \theta_2$.
Thus, θ_1 for which $\Phi_{V^{(n)}}(\theta_1, \theta_2)$ is non-zero are $\eta(\chi + e_i)$ (where $e_1 \ldots e_n$ are weights of $V^{(n)}$).

- $\eta(\chi + e_i)(z_k^{(n)}) = \sum_{j=1}^{n} \chi_j^k + (\chi_i + 1)^k - \chi_i^k$,
- so, $\eta(\chi + e_i)(u) - \eta(\chi)(u) = e^{(\chi_i+1)u} - e^{\chi_i u}$.

Thus, taking the ultraproduct, we get that $\Phi_V(\theta_1, \theta_2)$ is nonzero only if $\theta_1(u) - \theta_2(u) = e^{(b+1)u} - e^{bu}$ for some $b \in \mathbb{C}$.

Ultraproduct phenomenon

It turns out that we can get any value of $b \in \mathbb{C}$!

Bimodules of finite K-type

Lemma

Suppose $Y_1, Y_2 \in \mathcal{HC}(\theta_1, \theta_2)$ and Y_2 has finite K-type.
Then $\dim \mathrm{Hom}_{U^2}(Y_1, Y_2) < \infty$.

Proof.

Y_1 is a quotient of $\Phi_X(\theta_1, \theta_2)$ for some X. Thus,
$\dim \mathrm{Hom}_{U^2}(Y_1, Y_2) \leq \dim \mathrm{Hom}_{U^2}(\Phi_X(\theta_1, \theta_2), Y_2) =$
$\dim \mathrm{Hom}_{U^2}(\Phi_X, Y_2) = \dim \mathrm{Hom}(X, Y_2^{\mathrm{ad}}) < \infty$. □

Question

How do we construct examples of bimodules of finite K-type?

Example

Let $(\lambda, \lambda'), (\mu, \mu')$ be bipartitions, then $V_{\lambda,\lambda'} \otimes V^*_{\mu,\mu'}$ is a Harish-Chandra bimodule of finite K-type.

Bimodules of finite K-type

We would like to take filtered ultraproducts of Harish-Chandra bimodules.

That is, if $Y^{(n)}$ are Harish-Chandra bimodules for GL_n and $F^kY^{(n)}$ is a filtration by finite-dimensional $\mathfrak{k}$-modules that agrees with the PBW filtration on U^2 we can define
$F^kY = \prod_{\mathcal{F}} F^kY^{(n)}$, $Y := \bigcup F^kY$.

Warning

A priori F^kY lies in $\prod_{\mathcal{F}} \operatorname{Rep} GL(n, \mathbb{C})$, but **not** in $\operatorname{Rep}(GL_t)$. We need to require that for any bipartition (ν, ν')
$[F^kY^{(n)} : V^{(n)}_{[\nu,\nu']_n}]$ is constant for almost all n.

Lemma

Y as above has finite K-type if and only if $[Y^{(n)} : V^{(n)}_{[\nu,\nu']_n}]$ is constant for almost all n.

Bimodules of finite K-type

It is hard to compute $[Y^{\mathrm{ad}} : X]$ in general. However, we know something about finite-dimensional bimodules $\mathrm{Hom}_{\bar{\mathbb{Q}}}(V_{\mu^{(n)}}, V_{\lambda^{(n)}})$.

Lemma

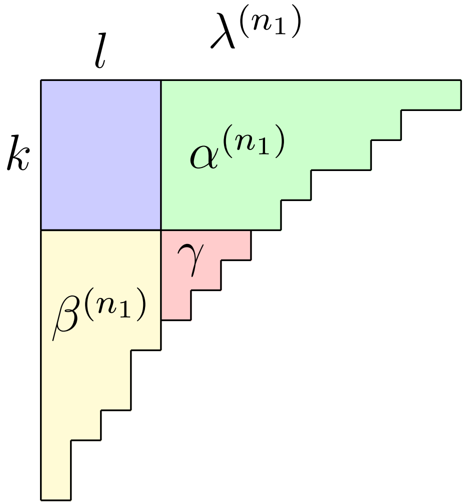
Under some assumptions on the sequences $\lambda^{(n)}, \mu^{(n)}$, for each bipartition (ν, ν') the multiplicity $[\mathrm{Hom}_{\bar{\mathbb{Q}}}(V_{\mu^{(n)}}, V_{\lambda^{(n)}}) : V_{[\nu,\nu']_n}]$ is constant for almost all n.

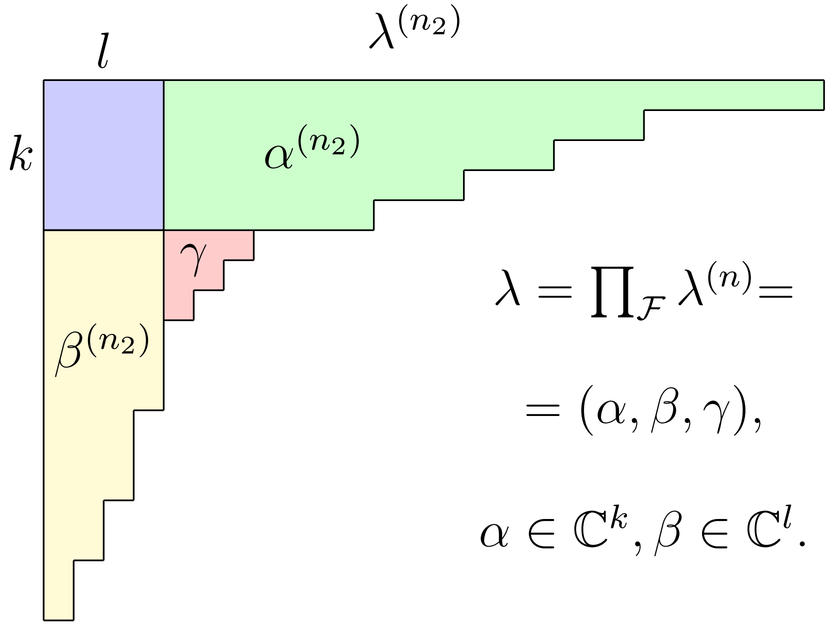
Corollary

Define $\underline{\mathrm{Hom}}(\mu, \lambda)$ to be the filtered ultraproduct of $\mathrm{Hom}_{\bar{\mathbb{Q}}}(V_{\mu^{(n)}}, V_{\lambda^{(n)}})$. Then it is a bimodule of finite K-type.

- The bimodules $\underline{\mathrm{Hom}}(\mu, \lambda)$ are simple.
- $V \otimes \underline{\mathrm{Hom}}(\mu, \lambda) = \bigoplus \underline{\mathrm{Hom}}(\mu, \lambda')$ and similarly, $V^* \otimes \underline{\mathrm{Hom}}(\mu, \lambda) = \bigoplus \underline{\mathrm{Hom}}(\mu, \lambda'')$, where λ', λ'' run over some finite sets.

Appendix





$$\lambda = \prod_{\mathcal{F}} \lambda^{(n)} =$$

$$= (\alpha, \beta, \gamma),$$

$$\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^l.$$

Roughly, we define the sequence $\lambda^{(n)}$ in the way that $k, l, \gamma(\lambda)$ are constant (or equivalently, universally bounded) and $\alpha^{(n)}(\lambda)$ and $\beta^{(n)}(\lambda)$ are unbounded parts.

And $\mu^{(n)}$ has the same k, l, some $\gamma(\mu)$ and $\alpha^{(n)}(\mu) - \alpha^{(n)}(\lambda)$ is a constant element in $\mathbb{Z}^k$ for almost all n. Similarly for β.

Thank you!