Periodic pencils of flat connections and their p-curvature

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arxiv: 2401.00636
arxiv: 2401.05652

1. Let $X$ be a smooth variety $/k = \overline{k}$, $X \times V \to X$ a trivial vector bundle on $X$.

Def. A polynomial family of flat connections on $X$ with fiber $V$ is a family of flat connections $\nabla(s) = d - B(s_1, \ldots, s_n)$, where $B \in \mathfrak{S}(X) \otimes \text{End} V$ depends polynomially on $s_1, \ldots, s_n$; i.e.

$\nabla(s) = d - B s_1 B_1 + \cdots + s_n B_n$, i.e. the dependence of $B$ on $s_i$ is linear homogeneous.

So the flatness conditions on $B_i$ are
\[ \partial B_i = 0, \quad [B_i, B_j] = 0 \]

(if \( \dim X = 1 \), these conditions are empty).

Def. A family \( \nabla(\vec{s}) \) is said to be periodic if there exist shift operators \( A_j(\vec{s}) \in \mathbb{GL}(V)(k(\vec{s})[x]) \) such that

\[ \nabla(\vec{s} + \vec{e}_j) \cdot A_j(\vec{s}) = A_j(\vec{s}) \cdot \nabla(\vec{s}). \]

It turns out that such pencils have many remarkable properties, and at the same time there are many interesting examples.

2. Let \( k = \mathbb{C} \). Fix \( x_0 \in X \) and let \( \rho_{\vec{s}} : \pi_1(X, x_0) \to \mathbb{GL}(V) \) be the monodromy representation of \( \nabla(\vec{s}) \).

Definition. A family \( \nabla(\vec{s}) \) has periodic monodromy if
Theorem 1. A pencil $\nabla(\mathfrak{g})$ has periodic monodromy if and only if its monodromy representation is defined over a finite Galois extension of $\mathbb{C}(\bar{q})$ and is Galois stable, where $\bar{q} = (q, \ldots, q_n)$,

$$q_j = e^{2\pi i \xi_j}.$$

Sketch of proof. We have a holomorphic map $\bar{\varphi} : \mathbb{C}^n \to \text{Hom}(\mathbb{T}_1(X, x_0), \text{GL}(V)) \bigg/ \text{GL}(V)$.

This map is $\mathbb{Z}^n$-periodic,

so we get a holomorphic map

$$\bar{\varphi} : (\mathbb{C}^\times)^n = \mathbb{C}^n / \mathbb{Z}^n \to \text{Hom}(\mathbb{T}_1(X, x_0), \text{GL}(V)) / \text{GL}(V).$$
Since $D$ is a pencil, $p$ is exponentially bounded, so $\tilde{f}$ has polynomial growth. Thus $\tilde{f}$ is a regular algebraic map. This map has an étale slice $\tilde{f} : (\mathbb{C}^*)^n \to \text{Hom}(\pi, (X,x_0), GL(V))$ where $(\mathbb{C}^*)^n$ is a finite cover of an open subset of $(\mathbb{C}^*)^n$.

Theorem 2. A family $V(s)$ with regular singularities is periodic if and only if it has periodic monodromy.

Sketch of proof. $\Rightarrow$ is obvious, so only need to prove $\Leftarrow$. If $V$ has RS and periodic monodromy, by the Riemann–Hilbert correspondence
For sufficiently generic \( \overline{s} \) there is an isomorphism \( A_j(\overline{s}) : \mathcal{V}(\overline{s} + \overline{e}_j) \cong \mathcal{V}(\overline{s}) \).

Pick a basis \( f_1, f_2, \ldots \) of \( k[x] \). There exists \( d \) such that for a large set of \( \overline{s} \),

\[
A_j(\overline{s}_x) = \sum_{i=1}^{d} A_{ji}(\overline{s}) f_i(x).
\]

Then the condition that \( A_j \) is an isomorphism \( \mathcal{V}(\overline{s}) \cong \mathcal{V}(\overline{s} + \overline{e}_j) \)
is a finite system of linear equations on \( A_{ji} \), which has an invertible solution for a large set of \( \overline{s} \). Then by elimination of quantifiers it has a solution over \( k(\overline{s}) \).

\[\square\]

(3) Theorem 2 allows us to give many examples of periodic pencils.
Example 1. KZ connections. Let $g$ be a simple f.d. Lie algebra, $\mathfrak{h} \subset g$ Cartan, $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ weights, $\beta \in \mathbb{Q}_+$, and consider the space

$$V = (M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_r}) \left[ \lambda_1 + \cdots + \lambda_r - \beta \right].$$

(this space depends only on $\beta$, not on $\lambda_1, \ldots, \lambda_r$). The KZ connection is the connection on $C^\infty$ diagonals

$$\nabla_{KZ} = \partial - \hbar \sum_{i=1}^{r} \left( \sum_{j \neq i} \frac{\partial_{i,j}}{z_i - z_j} \right) dz_i,$$

where $S \in (S^2 g)^g$ is the Casimir tensor. This connection is flat, and it is a pencil with parameters $S = (\hbar, \hbar \lambda_1, \ldots, \hbar \lambda_r) \ (1 + r \cdot \text{rank}(g)$ parameters).

Prop 3. $\nabla_{KZ}$ is periodic (up to rescaling by an integer).
Proof. Since $\nabla_{KZ}$ has regular singularities, it suffices to show that it has periodic monodromy, which follows from the Drinfeld–Kohno theorem:
The monodromy given by $R$-matrices for the quantum group $U_q(\mathfrak{g})$, which depend on $q = e^{\pi i t}$ (in the simply laced case) and $q^{\lambda_i}$.

Generalizations: KZ for Kac–Moody algebras, Lie superalgebras, trigonometric $KZ$, elliptic $KZ$, KZ in Deligne category, etc.

Example 2. Casimir connections. The Casimir connection is a connection on $\text{reg} = \mathfrak{g}$ with
fiber \( V = M_\lambda \sum (\lambda - \beta) \), given by

\[
\nabla_{\text{Cas}} = d - \frac{\hbar}{2} \sum_{\alpha \in \mathfrak{R}^+} \frac{e_\alpha f_\alpha + f_\alpha e_\alpha}{2} \frac{d\alpha}{\alpha}
\]

where \( e_\alpha, f_\alpha \) are root elements of \( \mathfrak{g} \). It is flat and forms a pencil with parameters \( \tilde{\Sigma} = (\hbar, \hbar \lambda) \).

Prop. 4. \( \nabla_{\text{Cas}} \) is a periodic pencil (up to rescaling \( \tilde{\Sigma} \)).

Proof. Since \( \nabla_{\text{Cas}} \) has RS, it suffices to show that it has periodic monodromy. But this follows from the theorem of A. Appel and V. Toledano-Laredo that the monodromy of \( \nabla_{\text{Cas}} \) is given by the quantum Weyl group.
of $U_q(g)$, which depends on $q = e^{2\pi i \hbar}$ and $q^\lambda$.

Generalizations: Casimir connections for Kac-Moody algebras, trigonometric Casimir connections.

Example 3. Dunkl connections: $W$-finite Weyl group with reflection representation $\gamma$, $V$ a representation of $W$. The Dunkl connection is the connection on $\mathfrak{g} \otimes \mathbb{C}^\lambda$ with fiber $V$:

$$D_{\text{Dunkl}} = d - \hbar \sum_{s \in \text{Reflections}(W)} s \cdot \frac{\partial s}{\partial s}$$

This is a flat pencil with parameter $\hbar$.

Prop. 5. $D_{\text{Dunkl}}$ is a periodic pencil.
proof. It has RS, so only need to check periodic monodromy. But this follows from the fact that the monodromy of the Dunkle connection is given by the Hecke algebra $H_q, q = e^{2\pi i t}$. (Ginzburg, Guay, Opdam, Rouquier).

Generalizations: Dunkl connections for complex reflection groups, trigonometric Dunkl connections.

Example 4. Let $\tilde{X} \to X$ be a conical symplectic resolution of singularities with finitely many torus fixed points. Consider the quantum connection $D_X$ with base $H^2(X) \setminus$ divisor and fiber
$H^*(X)$. This is a flat family depending on equivariant parameters $s_1, ..., s_n$ of the torus $T = (\mathbb{C}^*)^n$ acting on $X$, and it is known to be a pencil (in a certain basis called the stable basis).

Prop 6. $\nabla_x$ is a periodic pencil.

Proof. In the work of Okounkov, Pandharipande, Maulik, Braverman, it is shown that $\nabla$ shift operators coming from geometry (Stable envelopes).

4. Singularities of periodic pencils. One of the important properties of periodic pencils is that their
Singularities occur on hyperplanes defined over $\mathbb{Q}$ (at least up to shift). This echoes the fact that in representation theory, singularities tend to occur on such hyperplanes, as many examples of periodic pencils arise from representation theory (as we saw above). For example, let $B_j \subset \mathbb{C}^n$ be the set of points where $\nabla(\mathbf{s}) \neq \nabla(\mathbf{s}^2 + \mathbf{e}_j)$, and $\overline{B}_j$ be its Zariski closure. Then $\overline{B}_j$ is contained in the pole divisor of $A_j$.

Theorem 7. Every codimension 1 irreducible component of $\overline{B}_j$ is a hyperplane defined
over $\mathbb{Q}$ up to shift. Moreover, if $D(\Sigma)$ is generically irreducible, then poles of $\overline{A_j} \in \text{PGL}(V)$ (projection of $A_j$) occur on such hyperplanes.

Similar results occur for other types of singularities of $D(\Sigma)$, such as non-semisimplicity loci, jumping loci of endomorphism algebras, etc.

5. To prove Theorem 7, we will use the following theorem, which is interesting in its own right.

Theorem 8. Every periodic pencil has regular singularities.

This can be proved using
$p$-curvature as we will explain below.

Now the proof of Theorem 7 can be obtained from the following theorem of James Ax.

Theorem 9, (J. Ax, 1971). Let $Y \subset \mathbb{C}^n$ be an irreducible algebraic hypersurface such that $\exp(Y) \subset (\mathbb{C}^*)^n$ is also contained in an algebraic hypersurface. Then $Y$ is a hyperplane defined over $\mathbb{Q}$, up to shift.

Proof of theorem 7: By Th. 8 and the RH correspondence,

$\nabla (\xi + \zeta) \neq \nabla (\xi) \iff symp(\xi) \neq symp(\xi + \zeta)$
But since \( p > 0 \) by Theorem 1 is defined over some finite extension of \( \mathbb{C}(e^{2\pi i z}) \), we see that \( \exp(B_i) \) is contained in an algebraic hypersurface \( \mathbb{X} \subseteq \mathbb{C}((\mathbb{C}^*)) \).

So by Theorem 9, \( B_i \) is a hyperplane defined over \( \mathbb{Q} \) up to shift.

Now consider flat connections in characteristic \( p > 0 \). In characteristic 0, if a connection is flat (= has zero curvature) then it has a full set of formal flat sections near every point. This is, however, false in
characteristic \( p \): e.g. the equation \( \frac{dy}{dt} = y \) has no \( \neq 0 \) formal solutions near 0 (the solution \( e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \) is defined only in char 0).

That’s because connections in char \( p \), besides usual curvature, have another kind of curvature defined by Grothendieck, called the \( p \)-curvature. Namely, if \( D_i \) are covariant derivative corresponding to coordinates \( x_1, \ldots, x_r \) on \( X \) then the \( p \)-curvature is the collection of operators

\[ C_i = D_i^p. \]
Claim. These operators belong to $\mathbb{R}[x] \otimes \text{End} V$ (i.e. there are no derivatives).

Example: for the above equation $D = \partial - 1$ so $c = D^0 = (\partial - 1)^0 = 2^0 - 1 = -1$.

Now, if you have a formal flat section $f$ then $D_i f = 0$ so $D_i^0 f = 0$, hence $C_i f = 0$.

So to have a fundamental set of formal solutions we must have $C_i = 0$ for all $i$.

More generally, for generic $\alpha \in X$ the dimension of the space of formal solutions over $k[[x^0, \ldots, x^p]]$ is the dimension
of the common kernel of $C_i(x)$ (and it is a free module with an algebraic basis).

**Theorem 10.** If $D(s^e) = d - \sum_{j=1}^{n} s^e_j B_j$ is a periodic pencil, then the $p$-curvature operators $C_i$ of $D(s^e)$ are isospectral to

$$\sum_{j=1}^{n} (s^e_j - s^p_j) B^c_j,$$

where $B^c_j$ is the Frobenius twist of $B_j$. (Basically it means that all matrix coefficients are raised to power $p$).

This is a striking fact.
because in general we can say very little about the $p$-curvature.

Proof. (for simplicity $\dim X=1$, $n=1$, so $\nabla(s) = \mathcal{D} - sB$).

Let $C(s)$ be the $p$-curvature of $\mathcal{D}$. Since $\mathcal{D}(s+1) \cong \mathcal{D}(s)$, we have that $C(s+1)$ is conjugate to $C(s)$.

Now let $b_i(s) = \text{Tr}_{\Lambda^i} C(s)$ (coefficients of the characteristic polynomial). Then

$$b_i(s+1) = b_i(s). \quad (*)$$

Also $C(s)\cdot p$ is a polynomial in $s$ of degree $p$, so
bi(s) is a polynomial of s of degree pi. Equation (*) implies that bi(s) = βi(s - sp)
where βi is a polynomial of degree i. But also C(0) = 0, so C(s) is divisible by s, hence bi(s) is divisible by si. Thus βi(0) = βi. It remains to compute βi. To this end look at the leading term of bi(s).
We have C(s) = -spBp + O(sp-1) so Bi = trΛi(Bp) = trΛi(B0). Thus C is isospectral
Corollary II. The \( p \)-curvature of a periodic pencil is nilpotent when \( s_i \in k \).  

Theorem 11 gives us information about the \( p \)-curvature of connections in Ex 1-4 (K2, Casimir, Dunkl, Quantum Con n) since they can be reduced to characteristic \( p \) for almost all \( p \).

**Def. (N. Katz)** A flat connection defined over \( \mathbb{Q} \) is globally nilpotent if its reduction to characteristic \( p \) has nilpotent \( p \)- curvature for
almost all $p$. 

**Theorem 12.** A periodic pencil over $\overline{Q}$ evaluated at rational values of $s_j$ is globally nilpotent.

This follows from Corollary 11.

**Theorem 13 (N. Katz, 1970).** Any globally nilpotent connection has regular singularities.

This gives a proof of Thm 8 that periodic pencils have regular singularities — it follows from Theorem 12 and Theorem 13.

**Remark.** A theorem of N. Katz
says that if a connection $\mathfrak{D}/\mathbb{Q}$ is geometric (i.e., semisimple and irreducible constituents are direct sumsmands of Gauss–Manin connections), then $D$ is globally nilpotent.

The converse (for semisimple connections) is the André–Bombieri–Dwork conjecture. This generalizes the Grothendieck–Katz conjecture that if $D$ has zero $p$-curvature upon reduction to almost all primes then it has finite monodromy (= algebraic fundamental solution).

We don’t know if
all periodic pencils are geometric; many of them are (such as KZ, Casimir) but for some it is not known despite many efforts (e.g. Dunkl connections for exceptional groups). So these are explicit examples for which the ABD conjecture is open.

7. Def A regular flat connection $D$ defined over $\bar{Q}$ is quasi-geometric if its monodromy representation is also defined over $\bar{Q}$.

The pencils in Examples 1-4 evaluated at $\bar{s} \in \bar{Q}^n$ are
quasigeometric. Also every geometric connection is quasigeometric (since monodromy can be implemented by moving around cycles in the fiber of the family). The following theorem generalizes the monodromy theorem in Hodge theory.

**Theorem 14.** Every quasigeometric connection has quasiunipotent monodromy along divisors in a compactification $X \subset \overline{X}$.

**Proof.** Repeats Brieskorn's proof of the monodromy theorem, written down by Deligne. Let $\lambda$ be an eigenvalue of the residue of $D$ at
a codim = 1 component of 
\(X \setminus X\) in some trivialization. Then \(e^{2\pi i x}\) is an eigenvalue of the monodromy of \(D\) around \(D\). If \(D\) is quasi-geometric then both are algebraic, so by a theorem of Gelfond and Schneider (1934) \(\lambda \in \mathbb{Q}\), and \(e^{2\pi i x}\) is a root of 1.

**Conjecture 15.** If \(C\) is a braided fusion category then the regular connections on configuration spaces corresponding to its braid group representations (and
mapping class group representations if \( E \) is modular) are quasi-geometric.

**Theorem 16.** If \( E = \text{Rep} \, V \), where \( V \) is a strongly rational vertex algebra defined over \( \mathbb{Q} \) then Conjecture 15 holds for \( E \).

**Proof.** In this case the corresponding connection is the KZ connection of \( V \), which by definition is defined over \( \mathbb{Q} \). The monodromy of this connection is defined over \( \mathbb{Q} \) by Ocneanu rigidity.