Probabilities & Supergeometry

Measurement theory for dynamical discrete systems

Berkeley Math Seminar

Subho Chatterjee Cem Yetíşmíşoğlu Andrew Waldron

Center for Quantum Mathematics and Physics (QMAP), UC Davis

Department of Physics, Koç University, Turkey

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Questions

- Probabilities from exterior algebra?
- Geometric data?

Outline

- Precessing degrees of freedom
- Discrete systems
- Geometric model: symplectic vs odd symplectic
- Measurements
- Super phase-spacetime

A puzzle

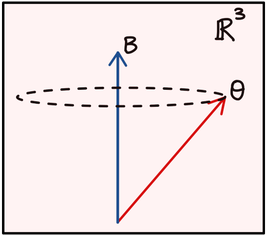
What is really spinning??

Exterior algebra, $\Lambda\mathbb{R}^3 = \frac{\mathbb{R}[\theta^1,\theta^2,\theta^3]}{(\theta^i\theta^j+\theta^j\theta^i)}$

Precession [Berezin and Marinov, 1977]

$$S[\theta^i(t)] = \int i\, dt \left(\frac{1}{2}\delta_{ij}\theta^i\dot{\theta}^j + B^i\epsilon_{ijk}\theta^j\theta^k \right)$$

Dynamics: $\dot{\vec{\theta}} = \vec{B} \times \vec{\theta}$



$\vec{\theta}$ precesses around $\vec{B}$

Measurements are probabilistic!

- Need **positive states** and **observables**

Problem

Anticommuting variables are not positive!

$$(\theta^i)^2 = 0, \quad \theta^i\theta^j = -\theta^j\theta^i$$

- **Solution**: supergeometry + quantum inspiration!

Discrete degrees of freedom

- **Discrete systems:** computer bits, coins, slot machines,. . .
- **Dynamics:** gate operations, tossing, spinning reels,. . .

Error-prone measurements $\Longrightarrow$ probabilistic dynamics

casino

E.g. Document editing

fat fingers

Bitstream:

$$\underbrace{110010010000111111011 \cdots 001110001101}_{m \text{ terms}}$$

2^m probabilities encoded by superfunction:

$$\phi(\theta) = \underbrace{\phi_I(p_I)}_{\text{probabilities}} \theta^I = \phi_0 + \phi_1\theta^1 + \cdots + \phi_{134}\theta^1\theta^3\theta^4 + \cdots + \phi^{12\cdots m}\theta_1\theta_2\cdots\theta_{\mathtt{m}-1}\theta_{\mathtt{m}}$$

$$\in \Gamma(\mathbb{R}^{0|m})$$

Covariance

Geometric models of dynamical systems and measurements?

Generalized dynamical phase-space

- *Symplectic structure, ω:*

$$d\omega = 0 \neq \omega^{\wedge n}, \quad \omega \in \Omega^2(M)$$

- *Symplectic vector field, ρ:*

$$\mathcal{L}_\rho \omega = 0, \quad \rho \in \Gamma(TM)$$

Definition

Dynamical phase-space $= (\underbrace{M^{2n}}_{\substack{\Cup \\ (p_i, q^i)}}, \omega, \rho)$

Phase-spacetime/Time covariance

Definition

Phase-spacetime $= (\underbrace{Z^{2n+1}}_{\substack{\Cup \\ (p_i, q^i, t)}}, \omega)$ [Herczeg et al., 2018, Casals et al., 2021]

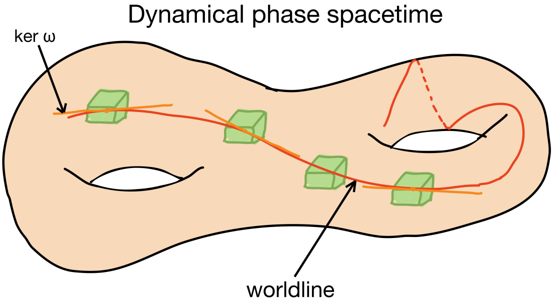


Figure: **dynamics** = unparameterized paths along $\ker \omega$

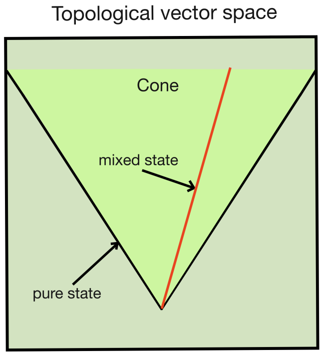
Measurements

- **States** = probability distributions = bounded non-negative functions

$$0 < \int \omega^{\wedge n} \Psi < \infty, \quad \Psi \in C^{\infty}(M)$$

- Liouville evolution:

$$\dot{\Psi} = \mathcal{L}_{\rho} \Psi$$



- **Observables** = dual to states = phase-space functions

- **Measurement** = Expectation value of some observable X in the state Ψ

$$\langle X \rangle_{\Psi} := \frac{\int_M \omega^{\wedge n} \Psi X}{\int_M \omega^{\wedge n} \Psi}$$

- **Superanalog?**

Phase-spacetime measurements

Definition

Laboratory = "time" slice = symplectic hypersurface $\Sigma \hookrightarrow Z^{2n+1}$

- instantaneous measurements
- measurement of the observable X at Σ in the state Ψ

$$\langle X \rangle_{\Sigma,\Psi} := \frac{\int_\Sigma \omega^{\wedge n} \Psi X}{\int_\Sigma \omega^{\wedge n} \Psi}$$

E.g. Symplectic foliation

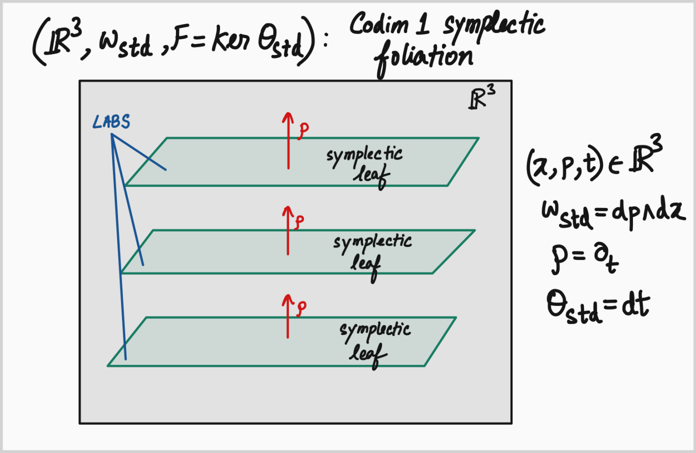


Figure: Family of labs

Positivity

Squaring trick

- states are positive $\Rightarrow$ squares

$$\Psi = \Phi^2$$

- squared superfunctions have positive bodies

$$(a + b\,\theta)^2 = a^2 + 2ab\,\theta$$

Problems:

- body is b independent
- $2ab$ does not have a definite sign ☹

- **Solution**: Define a new product!

$$\theta \star \theta = 1, \quad 1 \star \theta = \theta \star 1 = \theta, \quad 1 \star 1 = 1$$

- $\star$-squares:

$$(a + b\,\theta)^{\star 2} = a^2 + b^2 + 2ab\,\theta$$

$= (a^2 + b^2)\Big[p$  $(1+\theta) + p$  $(1-\theta)\Big]$

where p  $= \frac{(a+b)^2}{2(a^2+b^2)}$ and p  $= \frac{(a-b)^2}{2(a^2+b^2)}$ ☺ Probabilities!!

Wearing the quantum $\widehat{\text{hat}}$

Star product

$$\theta^i \star \theta^j = \delta^{ij} + \mathcal{O}(\theta^2), \quad \theta^i\theta^j \star \theta^k\theta^l = \frac{1}{2}(\delta^{jk}\delta^{il} - \delta^{ik}\delta^{jl}) + \mathcal{O}(\theta^2), \ \ldots$$

$$\star = \exp\left(\overleftarrow{\partial}_i \, \delta^{ij} \, \overrightarrow{\partial}_j\right)$$

$$\star = \text{Moyal!}$$

$$\text{body} \circ \star = \text{Hodge!}$$

- Clifford algebra representation

$$\sigma(F \star G) = \sigma(F)\sigma(G)$$

σ : superfunctions $\to$ Hermitian matrices

$$a + b\theta^1 + c\theta^2 - id\theta^1\theta^2 \stackrel{\sigma}{\mapsto} \begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix}$$

- Positivity of states = positivity of quantum mechanical trace

$$\int F \star F = \mathrm{Tr}\big(\sigma(F)^2\big) \propto a^2 + b^2 + c^2 + d^2$$

Supergeometry

Symplectic supermanifold

Definition

Super phase-spacetime $= (\mathcal{Z}^{2n+1|m}, \Omega)$
where

- $\mathcal{Z}^{2n+1|m} := (Z^{2n+1}, \mathcal{A}_m)$ and $\mathcal{A}_U \cong C^\infty(U) \otimes \Lambda\mathbb{R}^m$
- Ω is *symplectic* ($\dim_{\mathcal{A}_m} \ker \Omega = 1$)*, hermitian, Grassmann-even*.

Split structure

Short exact sequence of sheaves:

$$0 \to \mathcal{N} \to \mathcal{A} \xrightarrow{i^*} C^\infty Z \to 0$$

where $\mathcal{N}$ = sheaf of nilpotent ideals, $\mathcal{V}Z := \frac{\mathcal{N}}{\mathcal{N}^2}$, $\mathcal{E}Z := \Lambda\mathcal{V}Z$

- Real supermanifolds are **split** viz. there exists surjective morphism [Batchelor, 1979]

$$\pi : \mathcal{Z} \to (Z, C^\infty Z), \qquad \pi \circ i = \mathrm{Id}_Z$$

- Superaffine connection $\nabla : \Gamma(T\mathcal{Z}) \to \Gamma(T^*\mathcal{Z} \otimes T\mathcal{Z})$ canonically induces a $\mathbb{Z}$-**grading** [Koszul, 1994]

$$\nabla_X X = X, \quad (X - k\,\mathrm{Id}_{\mathcal{A}})\mathcal{N}^k \subset \mathcal{N}^{k+1}$$

Definition

Koszul bundle = representative exterior bundle = $(\mathcal{E}Z, \pi_\nabla)$

Induced $\star$-product

Theorem

Given a super phase-spacetime $(\mathcal{Z}, \Omega, \nabla)$ equipped with a flat, torsion-free, superaffine connection ∇ preserving Ω, there is a canonical star product $\star_\nabla$.

Proof

- Clifford map $\gamma : \Gamma(\mathrm{Ver}\mathcal{V}Z) \to \Gamma(\mathcal{C}Z)$ obeying

$$\{\gamma(u), \gamma(v)\} = 2\eta(u, v)\mathrm{Id}$$

- Quantization map $\sigma : \Gamma(\mathcal{E}Z) \to \Gamma(\mathrm{Ver}\mathcal{V}Z)$

$$\begin{aligned}\sigma(F) &= \gamma((\exp \eta^{-1}\nabla F)_0) \\ &= \mathrm{Id}F_0 + \gamma(\nabla F) + \frac{1}{2!}\gamma(\nabla^2 F) + \cdots + \frac{1}{m!}\gamma(\nabla^m F)\end{aligned}$$

- $\star$**-product**

$$\sigma(F \star_\nabla G) = \sigma(F)\sigma(G) \implies F \star_\nabla G = (\exp \eta^{-1})(F, G)$$

Supercone and measurements

Geometric data

$$(\mathcal{Z}, \Omega, \nabla_\Omega)$$

Definition

$\mathcal{C}$ = **Cone of states** $= \{\Psi \in \Gamma(\mathcal{E}^\nabla Z) | \Psi = \Phi \star_\nabla \Phi$ and $\Phi \in \mathcal{A}\}$

Definition

Measurement of observable $X \in \mathcal{A}$ in superlab Σ wrt state Ψ

$$\langle X \rangle_{\Sigma,\Psi} := \frac{\int_\Sigma \Psi \star X}{\int_\Sigma \Psi} = \frac{\int_\Sigma \Phi \star X \star \Phi}{\int_\Sigma \Phi \star \Phi}$$

classical looks like quantum!

- Statefunction

$$\Psi = \Phi^{\star 2}$$

- Canonical measure

$$dx^1 \dots dx^{2n} \sqrt{\det \Omega_{\rm body}} \circ {\rm body}$$

Examples

Supersymplectic measurement: Coin toss

- Supersymplectic structure:

$$\theta \in \mathcal{M} = \mathbb{R}^{0|1},\ \Omega = i\ d\theta \wedge d\theta$$

- (Normalized) State:

$$\begin{aligned}\Psi(\theta) &= p_{\text{tails}}(1+\theta) + p_{\text{heads}}(1-\theta), \quad p_{\text{tails}} + p_{\text{heads}} = 1 \\ &= (\phi_+ + \phi_-\theta)^{\star 2}\end{aligned}$$

where $\phi_\pm = \frac{1}{\sqrt{2}}\left(1 \pm \sqrt{1-(p_{\text{tails}} - p_{\text{heads}})^2}\right)^{1/2}$

- Observable $X = x_{\text{tails}}\left(\frac{1+\theta}{2}\right) + x_{\text{heads}}\left(\frac{1-\theta}{2}\right)$

$$\boxed{\langle X \rangle = \frac{\int_{\mathcal{M}} \Psi \star X}{\int_{\mathcal{M}} \Psi} = \frac{x_{\text{tails}}\, p_{\text{tails}} + x_{\text{heads}}\, p_{\text{heads}}}{p_{\text{tails}} + p_{\text{heads}}}}$$

Supersymplectic dynamics: 2-bit Markov

- Supersymplectic structure:

$$(t, \theta^1, \theta^2) \in \mathcal{M} = \mathbb{R}^{1|2}, \ \Omega = i(\delta_{ab} d\theta^a \wedge d\theta^b - H\epsilon_{ba}\theta^b d\theta^a \wedge dt)$$

- Dynamics: $\ker \Omega \ni \rho = \partial_t - \frac{H}{2}\epsilon_{ab}\theta^a \partial_{\theta^b}$
- State: $\Psi(t,\theta) = 1 + \psi_1\theta^1 + \psi_2\theta^2 + i\psi_{12}\theta^1\theta^2, \quad \psi_1^2 + \psi_2^2 + \psi_{12}^2 \leq 1$

$$= p_i(t)\xi^i$$

where ξ^i correspond to a bounding tetrahedron containing $\star$-squares.

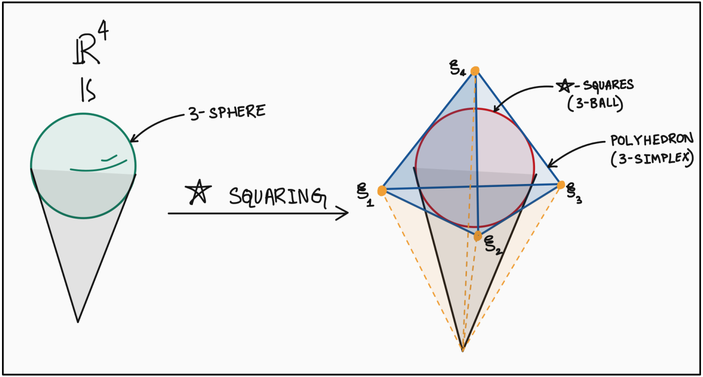


Figure: Geometry of $\star$-squares

Markov-like evolution

$$\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{pmatrix} = \begin{pmatrix} \frac{1+2\mathfrak{c}(t)}{3} & \frac{1-\mathfrak{c}(t)+\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1-\mathfrak{c}(t)-\sqrt{3}\mathfrak{s}(t)}{3} & 0 \\ \frac{1-\mathfrak{c}(t)-\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1+2\mathfrak{c}(t)}{3} & \frac{1-\mathfrak{c}(t)+\sqrt{3}\mathfrak{s}(t)}{3} & 0 \\ \frac{1-\mathfrak{c}(t)+\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1-\mathfrak{c}(t)-\sqrt{3}\mathfrak{s}(t)}{3} & \frac{1+2\mathfrak{c}(t)}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1(0) \\ p_2(0) \\ p_3(0) \\ p_4(0) \end{pmatrix}$$

where $\mathfrak{c}(t) := \cos\left(\frac{tH}{2}\right)$, $\mathfrak{s}(t) := \sin\left(\frac{tH}{2}\right)$

$$\Downarrow$$

$$p_1 + p_2 + p_3 = \text{const.}, \quad p_1^2 + p_2^2 + p_3^2 = \text{const.}$$

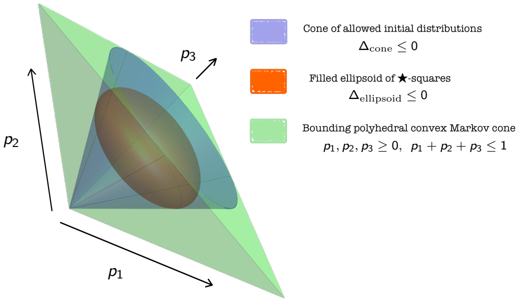


Figure: Dynamics = circular paths inside the orange ellipsoid

Questions & Future directions

- Quantization of Markov processes?
- Quantum Darboux theorem?
- Supersymmetries?
- Thermodynamic limit?
- Measurement theory for non-commuting algebras?

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