Representation homology and strong Macdonald conjectures

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1. Macdonald conjectures

In 1962, Freeman Dyson discovered the following combinatorial identity which he was unable to prove (see F.J. Dyson, '*Missed opportunities*'):

$$\mathrm{CT}\left\{\prod_{i\neq j}(1-x_ix_j^{-1})^k\right\} = \frac{(nk)!}{(k!)^n}\,, \quad k=0,\,1,\,2,\ldots$$

Here "CT" stands for the **constant term** of a Laurent polynomial.

G. E. Andrews (1975) proposed a q-generalization of the Dyson conjecture:

$$\mathrm{CT}\left\{\prod_{i\neq j}(\varepsilon_{ij}\,x_ix_j^{-1};\,q)_k\right\} = \frac{[nk]_q!}{([k]_q!)^n}\,, \quad k=0,\,1,\,2,\ldots$$

where $(x;q)_k := \prod_{l=1}^{k}(1-q^{l-1}x)$ and $\varepsilon_{ij}=1$ (for $i<j$) or q (for $i>j$).

In his 1982 paper '*Some conjectures for root systems*', I. G. Macdonald proposed a 'root system' generalization of the above conjectures.

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra (over $\mathbb{R}$ or $\mathbb{C}$), $\mathfrak{h} \subseteq \mathfrak{g}$ its Cartan subalgebra, $R \subset \mathfrak{h}^*$ a root system of $\mathfrak{g}$, $Q = Q(R)$, the root lattice (spanned by R). Write the elements of Q as formal exponentials e^{α} and define the **contant term map** $\mathrm{CT} : \mathbb{Z}[Q] \to \mathbb{Z}$, $\sum_{k \in \mathbb{Z}^l} a_k \, e^{k\alpha} \mapsto a_0$.

Macdonald conjectured that

$$\mathrm{CT} \left\{ \prod_{\alpha \in R_+} \prod_{i=1}^{k} (1 - q^{i-1} e^{-\alpha})(1 - q^i e^{\alpha}) \right\} = \prod_{i=1}^{l} \binom{kd_i}{k}_q \qquad (1)$$

where $l = \dim \mathfrak{h}$ and $d_1, d_2, \ldots, d_l$ are the fundamental degrees of R.

Macdonald also proposed a generalization of (1) to affine root systems by introducing a new *extra* parameter $t \in \mathbb{C}^*$. This became his famous

Constant Term (q,t)**-Conjecture:**

$$\frac{1}{|W|}\,\mathrm{CT}\left\{\prod_{n\geq 0}\prod_{\alpha\in R}\frac{1-q^n e^{\alpha}}{1-q^n t e^{\alpha}}\right\} = \prod_{n\geq 0}\prod_{i=1}^{l}\frac{(1-q^n t)(1-q^{n+1}t^{d_i-1})}{(1-q^{n+1})(1-q^n t^{d_i})} \qquad (2)$$

Note that specializing $t = q^k$ reduces (2) to (1).

In full generality, (2) was proved by Ivan Cherednik (1995), using representation theory of double affine Hecke algebras (DAHA).

Besides the above well-known conjectures, Macdonald's 1982 paper contained a number of less precise (and therefore less known) ones.

Let G be a compact connected Lie group associated to $\mathfrak{g}$. Macdonald observed that, using the classical Weyl integration formula, (1) can be written in the form

$$\int_G \prod_{j=1}^{k-1} \det(1 - q^j \operatorname{Ad} g)\, dg \ = \ \prod_{i=1}^{l} \prod_{j=1}^{k-1} (1 - q^{km_i + j}) \ ,$$

where $m_i = d_i - 1$ are the exponents of G and dg is the (normalized) Haar measure on G.

If we set $k = 2$ (and change $q \mapsto -q$), we get the well-known identity

$$\int_G \det(1 \ + \ q \operatorname{Ad} g)\, dg \ = \ \prod_{i=1}^{l} (1 + q^{2d_i - 1}) \ . \qquad (3)$$

The identity (3) arises from comparing the Poincarè series of both sides of the classical *Hopf-Koszul-Samelson Isomorphism*

$$\Lambda(\mathfrak{g})^G \cong \Lambda(\mathtt{Prim}\,\mathfrak{g}) \ ,$$

where $\mathtt{Prim}\,\mathfrak{g}$ is the subspace of primitive elements in the (Hopf) algebra of G-invariants in the exterior algebra of $\mathfrak{g}$.

Now, the identity (3) has a natural 'even' analogue:

$$\int_G \frac{dg}{\det(1 \ - \ q\,\mathrm{Ad}\,g)} \ = \ \prod_{i=1}^{l} \frac{1}{1 - q^{d_i}} \ , \qquad (4)$$

which comes from the well-known *Chevalley Isomorphism*

$$S(\mathfrak{g}^*)^G \ \cong \ S(\mathfrak{h}^*)^W \ .$$

Motivated by these observations Macdonald asked

Is there a (q,t)*-generalization of Identity* (4) *similar to* (2) ?

In joint work with G. Felder, A. Ramadoss, Th. Willwacher and S. Patotski, we have (apparently) answered Macdonald's question:

Conjecture ([BFPRW]):

$$\frac{(1-qt)^l}{(1-q)^l(1-t)^l}\,\mathrm{CT}\left\{\prod_{\alpha\in R}\frac{(1-qte^{\alpha})(1-e^{\alpha})}{(1-qe^{\alpha})(1-te^{\alpha})}\right\} = \sum_{w\in W}\frac{\det(1-qt\,w)}{\det(1-q\,w)\,\det(1-t\,w)}$$

where the determinants are taken in the natural representation of W on $\mathfrak{h}$.

The above identity can be written equivalently in the integral form:

$$\int_G \frac{\det(1\,-\,qt\,\mathrm{Ad}\,g)}{\det(1\,-\,q\,\mathrm{Ad}\,g)\,\det(1\,-\,t\,\mathrm{Ad}\,g)}\,dg \;=\; \frac{1}{|W|}\sum_{w\in W}\frac{\det(1-qtw)}{\det(1-qw)\,\det(1-tw)}$$

from which it is obvious that the specialization $t = 0$ yields (4).

Theorem ([BFPRW]). *The identity holds for* $\mathfrak{gl}_n$ *and* $\mathfrak{sl}_n$ *for all* n *.*

We now come to our main question:

Where do these identities come from?

We will answer this question — in fact, give a natural topological refinement of the above identities — in terms of a certain (non-abelian) homology theory of topological spaces, $\mathrm{HR}_*(X, G)$, called *representation homology*.

2. Representation varieties

Throughout k denotes a field of characteristic zero.

Let Γ be a (discrete) group, and let G be an affine algebraic group over k. The space $\mathrm{Rep}_G(\Gamma)$ of all representations of Γ in G is naturally an algebraic variety (more precisely, an affine k-scheme).

Examples

1. If $\Gamma = \mathbb{F}_n$ $(n \geq 1)$, then $\mathrm{Rep}_G(\Gamma) = G^n$.

2. If $\Gamma = \mathbb{Z}^2$, then $\mathrm{Rep}_G(\Gamma) \cong \{(x, y) \in G \times G \,|\, xy = yx\}$ is the classical G-commuting variety.

3. If X is a (based) topological space, then $\mathrm{Rep}_G(X) := \mathrm{Rep}_G[\pi_1(X)]$ is called the G-representation variety of X.

Applications

Representation varieties play an important role in many areas, most notably in representation theory and low-dimensional topology.

- **Representation theory.** $\mathrm{Rep}_G(\Gamma)$ carries a natural G-action. The G-orbits in $\mathrm{Rep}_G(\Gamma)$ are in natural bijection with the equivalence classes of representations of Γ in G. The geometry of G-orbits determines the algebraic structure of representations of Γ.

- **Topology.** One is usually interested in global algebro-geometric invariants defined in terms of representation varieties of fundamental groups. For example, if K is a knot in $\mathbb{S}^3$, many classical invariants of K arise from its character variety $\mathrm{Rep}_G(X_K)//G$, where $X_K := \mathbb{S}^3 \setminus K$.

  Examples include the Alexander polynomial ($G = k^*$), the A-polynomial, Chern-Simons invariant, Casson invariant, KBSM (for $G = \mathrm{SL}_2$), . . .

Representation varieties are very useful but there are some **problems**:

1. This kind of varieties are usually very singular, which makes it hard to understand their geometry. Thus, in representation theory, one faces the problem of resolving singularities of $\mathrm{Rep}_G(\Gamma)$.

2. In topology, the use of representation varieties is mostly limited to (compact oriented) surfaces, hyperbolic 3-manifolds and knot complements in $\mathbb{S}^3$, all of which are known to be *aspherical* spaces. The homotopy type of such a space is completely determined by the isomorphism type of its fundamental group, which makes representation varieties of these groups very strong and efficient invariants. For more general spaces, however, one needs to take into account a higher homotopy information, and looking at representation varieties of fundamental groups (or even, higher homotopy groups) is not enough. This raises the natural question:

What is a 'representation variety of a space' ?

3. Representation homology

Derived moduli spaces of G-local systems

Histrorically, the first answer to the above question was given by M. Kapranov (2001), and it was refined later by B. Toën and G. Vezzosi (2008) in their framework of derived (or homotopical) algebraic geometry.

Let G be an affine algebraic group defined over k. Given a pointed connected CW complex X consider the (framed) moduli space $\mathrm{Loc}_G(X,*)$ of G-local systems on X with trivialization at the basepoint of X.

This classical moduli space can be identified with the representation scheme $\mathrm{Rep}_G[\pi_1(X)]$, which, in turn, can be identified with the space $[X, BG]_*$ of homotopy classes of based maps from (a simplicial model of) X to the (simplicial) classifying space BG.

Kapranov constructed explicitly a simplicial DG scheme $\boldsymbol{R}BG$, which plays the role of 'injective resolution' of BG in the category of simplicial DG schemes. Then, replacing BG by $\boldsymbol{R}BG$, he defined an affine DG scheme

$$\mathbf{R}\mathrm{Loc}_G(X,*) := [X, \boldsymbol{R}BG]_* \ ,$$

which he called the *derived moduli space of* G*-local systems* on X

Toën and Vezzosi developed this construction in their HAG framework. Instead of working with simplicial DG schemes, HAG works in the category of *simplicial presheaves*, i.e. functors $\mathcal{F} : \mathtt{dAff}_k^{\mathrm{opp}} \to \mathtt{sSet}$, over the category of derived affine schemes, $\mathtt{dAff}_k := \mathtt{sComm}_k^{\mathrm{opp}}$, equipped with an appropriate model structure. The fibrant objects in this model category are called *derived stacks*. The space $\mathbf{R}\mathrm{Loc}_G(X,*)$ can be realized as a derived affine scheme (a special kind of derived stack) in this framework, which amounts to replacing Kapranov's explicit resolution $\boldsymbol{R}BG$ by an abstract fibrant resolution of BG in the category of derived stacks.

Derived representation functor

In joint work with A. Ramadoss and W.-K. Yeung [BRY], we developed a different approach to this problem that rests on a classical theorem of D. Kan in simplicial homotopy theory.

Recall that the affine scheme $\mathrm{Rep}_G(\Gamma)$ can be defined as the functor

$$\mathrm{Rep}_G(\Gamma): \ \mathtt{Comm\,Alg}_k \to \mathtt{Sets}\ , \quad A \mapsto \mathrm{Hom}_{\mathtt{Gr}}(\Gamma,\, G(A))\ .$$

This functor is representable, and we denote the corresponding commutative algebra by $\Gamma_G = \mathcal{O}[\mathrm{Rep}_G(\Gamma)]$.

Varying Γ (while keeping G fixed), we can now regard $\Gamma \mapsto \Gamma_G$ as a functor on the category of groups: $(\,-\,)_G : \mathtt{Gr} \to \mathtt{Comm\,Alg}_k$, which we call the *representation functor* in G.

The representation functor extends naturally to the category $\mathtt{sGr}$ of *simplicial* groups, taking values in simplicial commutative algebras:

$$(-)_G : \mathtt{sGr} \to \mathtt{sComm\,Alg}_k \ .$$

Both $\mathtt{sGr}$ and $\mathtt{sComm\,Alg}_k$ carry standard (projective) model structures, with weak equivalences being the weak homotopy equivalences of underlying simplicial sets.

The functor $(-)_G$ is *not* homotopy invariant: in general, it does not preserve weak equivalences and hence does not descend to a functor between $\mathtt{Ho}(\mathtt{sGr})$ and $\mathtt{Ho}(\mathtt{sComm\,Alg}_k)$. However, $(-)_G$ takes weak equivalences between cofibrant objects in $\mathtt{sGr}$ to weak equivalences in $\mathtt{sComm\,Alg}_k$. Hence, it has the (total) left derived functor

$$\boldsymbol{L}(-)_G : \ \mathtt{Ho}(\mathtt{sGr}) \to \mathtt{Ho}(\mathtt{sComm\,Alg}_k) \ .$$

We call $\boldsymbol{L}(-)_G$ the *derived representation functor* in G.

Heuristically, $\boldsymbol{L}(-)_G$ may be thought of as the "closest" universal approximation of the representation functor at the level of homotopy categories.

Recall that an affine algebraic group G is defined by its functor of points, which is a group-valued representable functor on commutative algebras. This functor extends in the natural way to simplicial commutative algebras:

$$G: \ \mathtt{sComm\,Alg}_k \to \mathtt{sGr}\ , \quad A_* \mapsto G(A_*)\ .$$

By definition, the classical representation functor $(-)_G$ is left adjoint to the functor of points of G, hence its simplicial extension is left adjoint to the above functor G. Thus, for any affine algebraic group, we have the adjunction

$$(-)_G\ :\ \mathtt{sGr} \rightleftarrows \mathtt{sComm\,Alg}_k\ :\ G\ . \tag{5}$$

Theorem ([BRY]). The functor G has a total right derived functor $\boldsymbol{R}G$: $\mathtt{Ho}(\mathtt{sComm\,Alg}_k) \to \mathtt{Ho}(\mathtt{sGr})$, which is right adjoint to $\boldsymbol{L}(-)_G$: thus, (5) induces the derived adjunction

$$\boldsymbol{L}(-)_G\,:\,\mathtt{Ho}(\mathtt{sGr}) \rightleftarrows \mathtt{Ho}(\mathtt{sComm\,Alg}_k)\,:\,\boldsymbol{R}G$$

Both $\boldsymbol{L}(-)_G$ and $\boldsymbol{R}G$ are absolute derived functors (in the sense of Kan).

This has one important implication.

Corollary. The derived representation functor $\boldsymbol{L}(-)_G$ preserves arbitrary homotopy colimits.

Next, recall that the model category $\mathtt{sGr}$ is Quillen equivalent to the category of *reduced* simplicial sets, $\mathtt{sSet}_0$, which is Quillen equivalent to the category $\mathtt{Top}_{0,*}$ of pointed connected topological spaces.

These classical equivalences are given by two pairs of adjoint functors

$$\mathbb{G} : \mathtt{sSet}_0 \rightleftarrows \mathtt{sGr} : \overline{W}\,, \qquad |-| : \mathtt{sSet}_0 \rightleftarrows \mathtt{Top}_{0,*} : \mathrm{Sing}$$

The functor $\mathbb{G}$ is given by the classical *Kan loop group* construction that assigns to a reduced simplicial set $X \in \mathtt{sSet}_0$ a semi-free simplicial group $\mathbb{G}X$, which is a simplicial model of the based loop space: $|\mathbb{G}X| \simeq \Omega|X|$.

The functor $\mathbb{G}$ preserves weak equivalences and hence induces a functor $\mathtt{Ho}(\mathtt{sSet}_0) \to \mathtt{Ho}(\mathtt{sGr})$. Combining $\mathbb{G}$ with the derived representation functor, we define the *representation homology* of a space $X \in \mathtt{sSet}_0$ by

$$\mathrm{HR}_*(X, G) := \pi_*[\boldsymbol{L}(\mathbb{G}X)_G]$$

Properties

1. $\mathrm{HR}_*(X, G)$ is a graded commutative algebra that depends only on the homotopy type of X and hence is a homotopy invariant of $|X|$.

2. $\mathrm{HR}_0(X, G) \cong (\pi_1(X))_G = \mathcal{O}[\mathrm{Rep}_G(\pi_1(X))]$. (To avoid confusion, we emphasize that $\mathrm{HR}_*(X, G) \ncong \mathrm{HR}_*(\pi_1(X), G)$ in general; however, if Γ is a discrete group and X is a $K(\Gamma, 1)$-space (*e.g.*, $X = \mathrm{B}\Gamma$), then we do have a natural isomorphism $\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(\Gamma, G)$.)

3. If G and H are two affine algebraic groups, then

$$\mathrm{HR}_*(X, G \times H) \cong \mathrm{HR}_*(X, G) \otimes \mathrm{HR}_*(X, H) \ .$$

4. It is a consequence of Theorem 1 that $\boldsymbol{L}(-)_G$ commutes with homotopy colimits. In particular, for two pointed spaces X and Y, we have

$$\mathrm{HR}_*(X \vee Y, G) \cong \mathrm{HR}_*(X, G) \otimes \mathrm{HR}_*(Y, G) \ .$$

Relation to derived algebraic geometry*

Toën and Vezzosi [HAG II] defined the derived stack $\mathbf{Map}(X, BG)$ classifying flat G-bundles on unpointed space X. This is one of the basic constructions in HAG. Using this construction, for any *pointed* space (simplicial set) X, we can then define

$$\mathbf{R}\mathrm{Loc}_G(X, *) := \mathrm{hofib}\,(\mathbf{Map}(X, BG) \;\to\; BG)\ ,$$

where the homotopy fibre is taken in the category of derived stacks. This generalizes Kapranov's original construction of derived moduli schemes of G-local systems on $(X, *)$ trivialized at $*$.

To compare with our construction, we associate to the derived representation functor the *derived representation scheme*

$$\mathrm{DRep}_G(X) := \mathbf{R}\mathrm{Spec}\,[\boldsymbol{L}(\mathbb{G}X)_G]\ .$$

Here '$\mathbf{R}\mathrm{Spec}$' stands for the derived Yoneda functor that assigns to a simplicial commutative algebra A — a derived ring in terminology of [HAG] — the simplicial presheaf (prestack)

$$\mathbf{R}\mathrm{Spec}(A) : \mathtt{dAff}_k^{\mathrm{op}} := \mathtt{sComm\,Alg}_k \ \to \ \mathtt{sSet}\ , \quad B \mapsto \underline{\mathrm{Hom}}(Q(A),\, B)\ ,$$

where $Q(A)$ is a cofibrant model for A and $\underline{\mathrm{Hom}}$ is the simplicial mapping space in $\mathtt{sComm\,Alg}_k$. For any $A \in \mathtt{sComm\,Alg}_k$, the prestack $\mathbf{R}\mathrm{Spec}(A)$ satisfies the descent condition for étale hypercoverings and hence defines a derived stack (which is a derived affine scheme in the sense [HAG]).

Theorem ([BRY'19]). For any pointed connected space X, there is an equivalence of derived stacks

$$\mathrm{DRep}_G(X) \simeq \mathbf{R}\mathrm{Loc}_G(X, *)\ .$$

4. Elementary construction

Classical homology

Recall that Δ denotes the simplicial category with objects

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}_+^{n+1} \,:\, x_0 + \ldots + x_n = 1\}\,,\; n \geq 0$$

For any space X, we can take a simplicial set model, which is represented by a functor $X_* : \Delta^{\mathrm{op}} \to \mathtt{Sets}$ (e.g. $X_* = \mathrm{Sing}_*(X) := \mathrm{Hom}_{\mathtt{Top}}(\Delta^*, X)$).

Now, given an abelian group A, the ordinary (simplicial) homology of X with coefficients in A is defined by

$$\mathrm{H}_*(X, A) := \pi_*[\Delta^{\mathrm{op}} \xrightarrow{X_*} \mathtt{Sets} \xrightarrow{A^{\oplus}} \mathtt{Ab}]$$

where $A^{\oplus}$ denotes the natural (direct sum) functor $S \mapsto A^{\oplus S} := \oplus_{s \in S} A$.

Similarly, for any commutative k-algebra R, one can define the (higher) *Hochschild homology* of X with coefficients in R by

$$\mathrm{HH}_*(X,R) := \pi_*[\Delta^{\mathrm{op}} \xrightarrow{X_*} \mathtt{Sets} \xrightarrow{R^{\otimes}} \mathtt{Vect}_k]$$

where $R^{\otimes}$ denotes the natural (tensor) functor $S \mapsto R^{\otimes S} := \otimes_{s\in S} R$.

Example. If $X = \mathbb{S}^1$ is a (simplicial) circle, then

$$\mathrm{HH}_*(\mathbb{S}^1, R) \cong \mathrm{HH}_*(R) \ ,$$

where $\mathrm{HH}_*(R)$ stands for the usual Hochschild homology of the algebra R.

Remark. The above definition of higher Hochschild homology is due to T. Pirashvili (2002). In topology, the tensor functor $(X,R) \mapsto R^{\otimes X}$ is usually referred to as the *Loday construction* on (X,R).

Representation homology

Let $\mathfrak{G} \subset \mathtt{Gr}$ denote the (small) category whose objects are the f.g. free groups $\langle n \rangle := \mathbb{F}_n$, one for each $n \geq 0$ and the morphisms are arbitrary group homomorphisms.

Any commutative Hopf algebra $\mathcal{H}$ gives a left $\mathfrak{G}$-module (still denoted $\mathcal{H}$)

$$\mathcal{H} : \ \mathfrak{G} \to \mathtt{Vect}_k \ , \quad \langle n \rangle \mapsto \mathcal{H}^{\otimes n}$$

Dually, any *cocommutative* Hopf algebra $\mathcal{K}$ gives a right $\mathfrak{G}$-module:

$$\mathcal{K} : \ \mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k \ , \quad \langle n \rangle \mapsto \mathcal{K}^{\otimes n}$$

Note that any left $\mathfrak{G}$-module $\mathcal{H}$ naturally extends to a functor on the category of all discrete groups: $\widetilde{\mathcal{H}} : \mathtt{Gr} \to \mathtt{Vect}_k$ (by taking the left Kan extension of $\mathcal{H}$ along the inclusion $\mathfrak{G} \hookrightarrow \mathtt{Gr}$).

Let $X \in \mathtt{sSet}_0$. Recall the Kan loop group $\mathbb{G}X$ of X is given by

$$(\mathbb{G}X)_n := \langle X_{n+1} \,|\, s_0(x) = 1\,,\, \forall\, x \in X_n \rangle = \langle X_{n+1} \setminus s_0(X_n) \rangle \ .$$

By construction, $\mathbb{G}X$ is functorial in X and $(\mathbb{G}X)_n \in \mathtt{Gr}$ is free for all n.

Now, given a commutative Hopf algebra $\mathcal{H}$, we define

$$\mathrm{HR}_*(X, \mathcal{H}) := \pi_*[\, \Delta^{\mathrm{op}} \xrightarrow{\mathbb{G}X} \mathtt{Gr} \xrightarrow{\widetilde{\mathcal{H}}} \mathtt{Vect}_k \,] \ .$$

Our key observation is the following

Proposition ([BRY]). $\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(X, \mathcal{O}(G))\,.$

This leads to a natural interpretation of representation homology in terms of classical (abelian) derived functors: namely, the derived tensor product functors $\mathrm{Tor}_*^{\mathfrak{G}}$ taken over the small category $\mathfrak{G}$.

Theorem ([BRY]). There is a natural homological spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^{\mathfrak{G}}(\underline{\mathrm{H}}_q(\Omega X; k), \mathcal{O}(G)) \implies \mathrm{HR}_{p+q}(X, G) \ .$$

relating the representation homology to the Pontryagin algebra of X.

This theorem has many interesting implications.

Corollary. Let Γ be a discrete group. Then

$$\mathrm{HR}_*(\mathrm{B}\Gamma, G) \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G))$$

In particular, $\mathcal{O}[\mathrm{Rep}_G(\Gamma)] \cong k[\Gamma] \otimes_{\mathfrak{G}} \mathcal{O}(G)$.

Corollary. Representation homology is a *rational* homotopy invariant for simply-connected spaces.

Relation to Hochschild homology

Theorem ([BRY]). For any (not necessarily pointed) topological space X,

$$\mathrm{HR}_*(\Sigma X_+, G) \cong \mathrm{HH}_*(X, \mathcal{O}(G))$$

Corollary. $\mathrm{HR}_*(\mathbb{S}^n, G) \cong \Lambda_k(\mathfrak{g}^*[n-1])$ for $n \geq 2$.

Remark. Despite many similarities between Hochschild and representation homology, there is one important difference: unlike $\mathrm{HH}_*(X, R)$, the $\mathrm{HR}_*(X, G)$ carries a natural algebraic G-*action* induced by the adjoint action of G. Examples show that this action depends on the space X in quite a nontrivial way, which makes representation homology a richer and more geometric theory than Hochschild homology.

6. Some computations and conjectures

Riemann surfaces.

Let Σ_g be a closed connected orientable surface of genus $g \geq 1$. There are natural isomorphisms

$$\mathrm{HR}_*(\Sigma_g, G) \;\cong\; \mathrm{Tor}_*^{\mathcal{O}(G)}(k, \mathcal{O}(G^{2g}))\ ,$$

where $\mathcal{O}(G^{2g})$ is viewed as an $\mathcal{O}(G)$-module via the algebra map α : $\mathcal{O}(G) \to \mathcal{O}(G^{2g})$: $f \mapsto \alpha(f)(x_1, y_1, \ldots, x_g, y_g) = f([x_1, y_1] \ldots [x_g, y_g])$.

Note that this implies that $\mathrm{HR}_i(\Sigma_g, G) = 0$ for all $i \geq \dim G$.

Conjecture 1. If G is a complex reductive group, we have

(a) If $g = 1$, then $\mathrm{HR}_i(\Sigma_g, G) = 0$ for all $i > \mathrm{rank}(G)$;

(b) If $g \geq 2$, then $\mathrm{HR}_i(\Sigma_g, G) = 0$ for all $i > \dim\ \mathcal{Z}(G)$.

Remark. Part (a) of Conjecture 1 holds for $G = \mathrm{GL}_n$ for all $n \geq 1$. Part (b) holds for the localized homology $\mathrm{HR}_*(\Sigma_g, G)_\varrho$ for a nonsingular $\varrho \in \mathrm{Rep}_G(\Sigma_g)$. In fact, for any such ϱ, we prove that $\mathrm{HR}_i(\Sigma_g, G)_\varrho = 0$ for

$$i > \dim_\varrho \mathrm{Rep}_G(\Sigma_g) - (1 - \chi(\Sigma_g)) \dim(G) = \dim \mathcal{Z}(G) \ ,$$

where the last equality follows from Goldman's formula.

Conjecture 2. Assume that G is one of the classical groups $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$, $n \geq 1$, or any simply-connected, reductive affine algebraic group. Then there is a natural isomorphism

$$\mathrm{HR}_*(\mathbb{T}^2, G)^G \cong [\mathcal{O}(T \times T) \otimes \Lambda_k^*(\mathfrak{h}^*)]^W \ .$$

Remark. Conjecture 2 has been recently proved (or announced to be proved) for $G = \mathrm{GL}_n$ by D. Gaitsgory and T. Feng (Oct. 2023) and Pengui Li, D. Nadler and Zhiwei Yun (Jan. 2023).

Link complements. Let $X := \mathbb{R}^3 \backslash L$ denote the complement of (a regular neighborhood) of a (smooth) link L in $\mathbb{R}^3$. Recall two classical facts:

Fact (Alexander). Every link L can be obtained as the closure of a (not necessarily unique) braid $\beta \in B_n$ in $\mathbb{R}^3$. (We write $L = \hat{\beta}$ in this case).

Fact (Artin). The braid group B_n has a faithful representation in $\mathrm{Aut}(\mathbb{F}_n)$ given by

$$\sigma_i \,:\, \begin{cases} x_i & \mapsto \quad x_i\, x_{i+1}\, x_i^{-1} \\ x_{i+1} & \mapsto \quad x_i \\ x_j & \mapsto \quad x_j \quad (j \neq i, i+1) \end{cases}$$

Proposition. For any link $L = \hat{\beta}$ in $\mathbb{R}^3$, we have

$$\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G) \,\cong\, \mathrm{HH}_*(\mathcal{O}(G^n), \mathcal{O}(G^n)_\beta) \ .$$

where the bimodule $\mathcal{O}(G^n)_\beta$ is twisted via the Artin representation of B_n.

7. Representation homology of simply-connected spaces

Assume $k = \mathbb{Q}$. Let X be simply-connected of finite rational type. There is a commutative *cochain* dg algebra $\mathcal{A}_X$ (called the *Sullivan model* of X) and a chain dg Lie algebra $\mathcal{L}_X$ (called the *Quillen model* of X).

Both models encode the rational homotopy of X. $\mathrm{H}^*(\mathcal{A}_X) \cong \mathrm{H}^*(X, \mathbb{Q})$ while $\mathrm{H}_*(\mathcal{L}_X) \cong \pi_*(\Omega X)_{\mathbb{Q}}$. The two models are related by a quasi-isomorphism

$$\mathcal{C}^*(\mathcal{L}_X, \mathbb{Q}) \xrightarrow{\sim} \mathcal{A}_X \ .$$

Main Theorem ([BRY]). For any simply-connected space X of finite rational type, there are natural isomorphisms

$$\mathrm{HR}_*(X, G) \cong \mathrm{H}^{-*}(\mathfrak{g}(\overline{\mathcal{A}}_X); \mathbb{Q}) \,, \qquad \mathrm{HR}_*(X, G)^G \cong \mathrm{H}^{-*}(\mathfrak{g}(\mathcal{A}_X), \mathfrak{g}; \mathbb{Q}) \ .$$

Note that $\mathfrak{g}(\mathcal{A}_X) := \mathfrak{g} \otimes \mathcal{A}_X$ is a cochain dg Lie algebra.

8. Spaces with polynomial representation homology

Assume X is simply connected (thus, $\mathrm{HR}_0(X,G) = k$) and G is reductive. In this case, it is natural to treat $\mathrm{HR}_*(X,G)$ as an object of representation theory — or even classical invariant theory — and ask basic questions about the structure of the algebra $\mathrm{HR}_*(X,G)$ as a G-module and its subalgebra $\mathrm{HR}_*(X,G)^G$ of G-invariants.

Perhaps, the first basic question is:

Question 1. *When is the invariant algebra* $\mathrm{HR}_*(X,G)^G$ *free and finitely generated (i.e. isomorphic to the graded symmetric algebra of a (locally) finite-dimensional graded vector space over k)*?

This question turns out to be closely related to Macdonald conjectures!

Topological character maps

Let $\mathcal{L}X := \mathrm{Map}(\mathbb{S}^1, X)$ denote the free loop space of X. The natural $\mathbb{S}^1$-action on $\mathcal{L}X$ allows one to define the Frobenius (Adams) operations

$$\Psi^n : \mathrm{H}_*^{\mathbb{S}^1}(\mathcal{L}X; k) \to \mathrm{H}_*^{\mathbb{S}^1}(\mathcal{L}X; k) \ .$$

which are induced by the n-fold coverings $\mathbb{S}^1 \to \mathbb{S}^1$, $e^{i\varphi} \mapsto e^{in\varphi}$.

Let $\mathrm{H}_*^{\mathbb{S}^1,(p)}(\mathcal{L}X; k)$ denote the common eigenspace of the Ψ^n's with eigenvalues n^p for all $n \geq 0$. A theorem of Burghelea, Fiedorowicz and Gajda asserts that, if all (rational) Betti numbers of X are finite, then each Hodge component of $\mathrm{H}_*^{\mathbb{S}^1}(\mathcal{L}X,\, k)$ is locally finite: i.e.,

$$\dim_k \mathrm{H}_i^{\mathbb{S}^1,\,(p)}(\mathcal{L}X,\, k) < \infty \quad \text{for all } i \geq 0 \text{ and all } p \geq 0 \ .$$

Then, for $P \in \mathrm{Sym}^d(\mathfrak{g}^*)^G$, there are natural maps of graded vector spaces

$$\mathrm{Tr}_{\mathfrak{g}}^P \ : \ \mathrm{H}_*^{\mathbb{S}^1,\,(d-1)}(\mathcal{L}X;k) \to \mathrm{HR}_*(X,G)^G \ .$$

We call these maps *Drinfeld traces*.

Now, let G be complex reductive. Recall that $\mathrm{Sym}(\mathfrak{g}^*)^G \cong \mathbb{C}[P_1,\ldots,P_l]$, where $P_1,\ldots,P_l$ are homogeneous polynomials of degrees $d_1,\ldots,d_l$ (the fundamental degrees of $\mathfrak{g}$).

The maps $\mathrm{Tr}_{\mathfrak{g}}^{P_i}, i = 1,\ldots,l$ assemble into a homomorphism of graded commutative algebras

$$\mathrm{Tr}_{\mathfrak{g}}(X) \ : \ \Lambda_k[\oplus_{i=1}^l \mathrm{H}_*^{\mathbb{S}^1,(m_i)}(\mathcal{L}X;\mathbb{C})] \to \mathrm{HR}_*(X,G)^G \,,$$

where $m_i := d_i - 1$ are the exponents of G. We call this map the *Drinfeld homomorphism*.

Example 1. Let $X = \mathbb{S}^{2n}$, $n \geq 1$. The Drinfeld homomorphism for X in this case (as a homomorphism of $\mathbb{Z}_2$-graded algebras) coincides with the classical Hopf-Koszul-Samelson Isomorphism

$$\Lambda[\mathtt{Prim}(\mathfrak{g})] \cong \Lambda(\mathfrak{g}^*)^G$$

Example 2. Let $X = \mathbb{S}^{2n+1}$, $n \geq 1$. The Drinfeld homomorphism in this case (as a homomorphism of $\mathbb{Z}_2$-graded algebras) coincides with the inverse of the Chevalley Restriction Isomorphism:

$$\mathrm{Sym}(\mathfrak{g}^*)^G \cong \mathrm{Sym}(\mathfrak{h}^*)^W \ .$$

Example 3. Let $G = (\mathbb{C}^*)^l$ be an algebraic torus, then $m_i = 0$ for all $i = 1, 2, \ldots, l$, and $\mathrm{HR}_*(X, G)^G = \mathrm{HR}_*(X, G)$, because G is commutative. On the other hand, for any X, we have $\mathrm{H}_*^{S^1,\,(0)}(\mathcal{L}X,\, \mathbb{C}) \cong \mathrm{H}_{*+1}(X,\, \mathbb{C})$, where the isomorphism is induced by the classical Gysin map $\mathrm{H}_*^{S^1}(\mathcal{L}X,\, \mathbb{C}) \to \mathrm{H}_{*+1}(\mathcal{L}X,\, \mathbb{C})$. Thus, for an algebraic torus, the Drinfeld homomorphism becomes

$$\Lambda\big[\mathrm{H}_{*+1}(X;k)^{\oplus l}\big] \;\to\; \mathrm{HR}_*(X,G)\ .$$

A simple calculation shows that the above map is an *isomorphism* for any simply connected space X and, in fact, for any commutative — not necessarily diagonalizable — algebraic group G.

The above examples motivate the following

Question 2. *For which spaces X, is the map $\mathrm{Tr}_{\mathfrak{g}}(X)$ an isomorphism*?

The following theorem gives a simple answer in terms of the rational cohomology algebra of X.

Theorem ([BRY]). *Assume that the algebra* $\mathrm{H}^*(X;\mathbb{Q})$ *is either generated by one element (in any dimension) or* freely *generated by two elements: one in even and one in odd dimensions. Then, the map* $\mathrm{Tr}_{\mathfrak{g}}(X)$ *is an isomorphism for any complex reductive algebraic group* G.

As a consequence, we get an answer to our Question 1:

Corollary. If X satisfies the conditions of the above theorem, then for any complex reductive group G, $\mathrm{HR}_*(X,G)^G$ is a free graded commutative algebra of locally finite type over $\mathbb{C}$.

The proof relies on Main Theorem and a certain (minor) refinement of the main results of S. Fishel, I. Grojnowski and C. Teleman in

[FGT] *The Strong Macdonald Conjecture and Hodge theory on the loop Grassmannian*, Ann. Math. **168** (2008), 175–220.

The above paper settles the so-called *Strong Macdonald Conjecture* (SMC) describing the structure of cohomology of nilpotent truncations of the current Lie algebra $\mathfrak{g}[z]$ for any complex reductive Lie algebra $\mathfrak{g}$ — a deep and celebrated result in representation theory proposed as a conjecture by I. Macdonald, B. Feigin, and P. Hanlon in the early 1990's and proved (in full generality) in [FGT].

The proof of SMC in [FGT] is an algebraic tour de force. Our theorem gives a topological meaning to this conjecture. It also suggests a natural "odd" analogue — the *'super' Strong Macdonald Conjecture* (sSMC), see below.

Macdonald Identities

Example 1. Let us consider the spaces X with rational cohomology algebra $\mathrm{H}^*(X,\mathbb{Q}) \cong \mathbb{Q}[z]/(z^{r+1})$, where the generator z is in even dimension $d \geq 2$. The most familiar examples of such spaces are the even-dimensional spheres $\mathbb{S}^{2n}$ ($r = 1$, $d = 2n$) and the complex projective spaces $\mathbb{CP}^r$ ($r \geq 1$, $d = 2$), the quaternionic projective spaces $\mathbb{HP}^r$ ($r \geq 1$, $d = 4$) and the Cayley plane $\mathbb{OP}^2$ ($r = 2$, $d = 8$). For these spaces, we have

$$\mathrm{HR}_*(X,G)^G \;\cong\; \Lambda\,[\xi_1^{(i)}, \xi_2^{(i)}, \ldots, \xi_r^{(i)} \;:\; i = 1,2,\ldots,l]\;,$$

where the generators $\xi_j^{(i)}$ have homological degree

$$\deg\,\xi_j^{(i)} \;=\; (d(r+1)-2)m_i + dj - 1\;.$$

Notice that, in this case, the algebra $\mathrm{HR}_*(X, G)^G$ is generated by finitely many elements of *odd* degrees: hence, it is finite-dimensional (as a vector space) and concentrated in finitely many homological degrees.

In fact, knowing the exact degrees of generators, it is easy to calculate the exact upper bound for the vanishing of $\mathrm{HR}_n(X, G)^G$:

$$\sum_{i=1}^{l} \sum_{j=1}^{r} \deg\, \xi_j^{(i)} \;=\; \frac{1}{2}\, r\,(d(r+1) - 2)\, \dim G\ .$$

Somewhat miraculously, this allows one to determine the exact upper bound for the *full* representation homology of X:

$$\mathrm{HR}_n(X, G) \;=\; 0 \quad \text{for all } \; n > \frac{1}{2}\, r\,(d(r+1) - 2)\, \dim G\ .$$

Now, the weighted Euler-Poincaré series of $\mathrm{HR}_*(X,G)^G$ is given by

$$P_{X,G}(q,z) \;=\; \prod_{i=1}^{l}\prod_{j=1}^{r}(1\,+\,q^{j+m_i(r+1)}\,z^{\deg\,\xi_j^{(i)}})\ ,$$

which specializes (at $z=-1$) to the (weighted) Euler characteristic

$$\chi_{X,G}(q) \;=\; \prod_{i=1}^{l}\prod_{j=1}^{r}(1\,-\,q^{j+m_i(r+1)})\ .$$

The latter can be also computed using our Main Theorem: as an Euler characteristic of the Chevalley-Eilenberg complex $\mathcal{C}^{-*}(\mathfrak{g}(\mathcal{A}_X),\mathfrak{g};\mathbb{C})$, where $\mathcal{A}_X$ is the (minimal) Sullivan model of the corresponding space X. The resulting equality of Euler characteristics gives the combinatorial identity

$$\mathrm{CT}\left\{\prod_{j=0}^{r}\prod_{\alpha\in R}(1-q^j e^{\alpha})\right\}=\prod_{i=1}^{l}\prod_{j=1}^{r}\frac{1-q^{j+m_i(r+1)}}{1-q^j},$$

which is precisely the Macdonald's CT q-identity (1).

Example 2. Let X be a simply connected space such that $\mathrm{H}^*(X;\mathbb{Q})\cong\mathbb{Q}[z,s]$, where $d=|z|$ is even and $p=|s|$ is odd. Examples include:

- $K(\mathbb{Z},d)\times\mathbb{S}^p$, where $d\geqslant 2$ is even and $p\geqslant 3$ is odd ($|z|=d$, $|s|=p$)
- $\mathbb{CP}^\infty\times\mathbb{S}^{2r+1}$ (rationally equivalent to $K(\mathbb{Z},2)\times\mathbb{S}^{2r+1}$)
- $\mathbb{HP}^\infty\times\mathbb{S}^{4r+3}$ (rationally equivalent to $K(\mathbb{Z},4)\times\mathbb{S}^{4r+3}$)

In this case, the algebra $\mathrm{HR}_*(X,G)^G$ can be also described explicitly. The combinatorial identity arising from Euler characteristics is two-variable:

$$\frac{1}{|W|}\,\mathrm{CT}\left\{\prod_{j=1}^{\infty}\prod_{\alpha\in R}\frac{1-q^{j-1}e^{\alpha}}{1-q^{j-1}\,t\,e^{\alpha}}\right\}=\prod_{i=1}^{l}\prod_{j=1}^{\infty}\frac{(1-q^{j-1}t)\,(1-q^{j}\,t^{m_i})}{(1-q^{j})\,(1-q^{j-1}\,t^{m_i+1})}\,.$$

This is precisely Macdonald's CT (q,t)-identity (2).

We now turn to our last question:

What is a topological meaning of the 'even' analogues of Macdonald identities?

8. The derived Chevalley homomorphism

Note that the "even" identities *cannot* be induced by the Drinfeld homomorphism for any space X: its RHS is not an Euler characteristic of a free graded commutative algebra. We need to construct a different map.

Let G be a connected reductive group with maximal torus $T \subset G$ and an associated Weyl agroup W. Then, for any space X, there is a natural restriction map

$$\Phi_G(X) : \ \mathrm{HR}_*(X, G)^G \to \mathrm{HR}_*(X, T)^W$$

which we call the *derived Chevalley homomorphism*.

Example. Let $X = \mathbb{S}^{2p+1}$ be any odd-dimensional sphere. In this case, the homomorphism $\Phi_G(X)$ can be identified with the classical Chevalley restriction map, and hence is an algebra isomorphism. At the level Euler characteristics, it gives the classical identity (4).

In general, we propose the following the 'super' (analog of) Strong Macdonald Conjecture (sSMC).

Conjecture. Assume that $\mathrm{H}^*(X;\mathbb{Q})$ is either generated by one element in odd dimension or *freely* generated by two elements in odd dimensions. Then the derived Chevalley homomorphism $\Phi_G(X)$ is an isomorphism, at least for G of classical type (GL_n, SL_n or Sp_{2n}).

The spaces that satisfy the above conditions are (1) the odd-dimensional spheres $\mathbb{S}^{2p+1}$ and (2) the products of two odd-dimensional spheres $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ $(p, q \geq 0)$.

In the first case, the above Conjecture follows from the classical Chevalley Restriction Theorem (see Example), while in the second (for $p, q > 0$), it reduces to the main conjecture [BFPRW] that still remains wide open.

In the special case $p = q = 0$, Conjecture yields an isomorphism:

$$\mathrm{HR}_*(\mathbb{S}^1 \times \mathbb{S}^1, G)^G \cong [\mathcal{O}(T \times T) \otimes \Lambda^*(\mathfrak{t}^*)]^W$$

where $\mathfrak{t}$ is the Lie algebra of T. In a recent preprint, P. Li, D. Nadler and Z. Yun established this last isomorphism (though under a technical assumption which has yet to be proven), using the newly developed representation-theoretic techniques (the so-called Betti Geometric Langlands).

For $p, q \geq 1$, the above Conjecture implies

$$\mathrm{HR}_*(\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1},\, G)^G \;\cong\; [\,\mathrm{Sym}_k(\mathfrak{h}^* \oplus \mathfrak{h}^*) \otimes \Lambda(\mathfrak{h}^*)\,]^W \;,$$

where the copies of $\mathfrak{h}^*$ in Sym have homological degree $2p$ and $2q$ and the copy of $\mathfrak{h}^*$ in Λ has homological degree $2(p+q)+1$.

For example, for $G = \mathrm{GL}_n$, the RHS is isomorphic to the ring of three-diagonal symmetric polynomials:

$$k[x_1, \ldots, x_n; y_1, \ldots, y_n; \theta_1, \ldots, \theta_n]^{S_n}$$

where

$$\deg(x_i) = 2p \;, \quad \deg(y_i) = 2q \text{ and } \deg(\theta_i) = 2(p+q)+1$$

for all $i = 1, \ldots, n$.

Theorem ([BRY]). For $X = \mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$, the derived Chevalley homomorphism $\Phi_G(X)$ induces the identity

$$\frac{(1-qt)^l}{(1-q)^l(1-t)^l}\,\mathrm{CT}\left\{\prod_{\alpha\in R}\frac{(1-qte^{\alpha})(1-e^{\alpha})}{(1-qe^{\alpha})(1-te^{\alpha})}\right\} = \sum_{w\in W}\frac{\det(1-qt\,w)}{\det(1-q\,w)\,\det(1-t\,w)}$$

This is precisely the ‘even’ analogue of the original CT (q,t)-Macdonald Identity discovered in [BFPRW] (see Section 1).

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