Math 202B Solutions

Assignment 9

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32. Let $\mathcal{A}$ be a σ-algebra on a set X.

    (a) Prove that if μ is a positive σ-finite measure on $\mathcal{A}$, then there is a finite measure on $\mathcal{A}$ that is mutually absolutely continuous with respect to μ.

    Proof: Since μ is σ-finite we can write $X = \bigcup_{n=1}^{\infty} E_n$, where $E_1, E_2, \ldots$ are disjoint sets in $\mathcal{A}$ and $0 < \mu(E_n) < \infty$ for all n. For each n define the measure μ_n by

    $$\mu_n = \frac{2^{-n}}{\mu(E_n)} \mu_{E_n}.$$

    Thus $\|\mu_n\| = 2^{-n}$, so $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$, implying that the series $\sum_{n=1}^{\infty} \mu_n$ converges in the normed space $M(\mathcal{A})$. (It is easy to prove that the sequence of partial sums is Cauchy.) Letting $\nu = \sum_{n=1}^{\infty} \mu_n$, we have $\mu \ll \nu \ll \mu$ (since a set is μ-null if and only if its intersection with each E_n is μ-null).

    (b) Let $\mu_1, \mu_2, \ldots$ be positive σ-finite measures on $\mathcal{A}$. Prove that there is a finite measure ν on $\mathcal{A}$ such that $\mu_n \ll \nu$ for all n.

    Proof: By (a), we can assume without loss of generality that each μ_n is a finite measure. Since μ_n and $c\mu_n$ are mutually absolutely continuous for any positive constant c, we can in addition assume without loss of generality that $\|\mu_n\| < 2^{-n}$ for each n. Then the series $\sum_{n=1}^{\infty} \mu_n$ converges in the norm of $M(\mathcal{A})$, say to ν, and it is obvious that $\mu_n \ll \nu$ for all n.

33. Let $\mathcal{A}$ be a σ-algebra on a set X. Let μ and ν be positive measures in $M(\mathcal{A})$ such that $\|\mu - \nu\| = \|\mu\| + \|\nu\|$. Prove that $\mu \perp \nu$.

    Proof 1: Let $\xi = \mu + \nu$. Since $\mu \ll \xi$ and $\nu \ll \xi$, by the Radon-Nikodym theorem, we have $\mu = f\xi$ and $\nu = g\xi$, where f and g are ξ-integrable functions, nonnegative since μ and ν are positive measures. Thus,

    $$\|\mu\| = \int f \, d\xi, \qquad \|\nu\| = \int g \, d\xi, \qquad \|\mu - \nu\| = \int |f - g| \, d\xi,$$

    so

    $$0 = \|\mu\| + \|\nu\| - \|\mu - \nu\| = \int (f + g - |f - g|) \, d\xi.$$

    As the integrand on the last integral is nonnegative, we have

    $$f + g - |f - g| = 0 \ \xi\text{-a.e.},$$

    which implies that $fg = 0$ a.e. Hence, if $A = \{f > 0\}$ and $B = X \setminus A$, then $\mu = \mu_A$ and $\nu = \nu_B$, showing that $\mu \perp \nu$.

    Proof 2: Let $X = A \sqcup B$ be the Hahn decomposition of $\mu - \nu$. Then

    $$\|\mu - \nu\| = (\mu - \nu)(A) - (\mu - \nu)(B) = \mu(A) + \nu(B) - \nu(A) - \mu(B),$$

    while

    $$\|\mu\| + \|\nu\| = \mu(A) + \nu(B) + \nu(A) + \mu(B).$$

    Therefore, $\nu(A) + \mu(B) = 0$; since $\nu(A), \mu(B) \geq 0$, this implies $\nu(A) = \mu(B) = 0$. Thus, $\mu = \mu_A$ and $\nu = \nu_B$.

34. Let μ and ν be measures in $M(\mathbb{R}^N)$ such that $\mu \ll \lambda_N$. Prove that $\mu * \nu \ll \lambda_N$.

    Proof: Let E be a Lebesgue-null Borel set. We have

    $$(\mu * \nu)(E) = \int \left(\int \chi_E(x + y) \, d\mu(x) \right) d\nu(y) = \int \mu(E - y) \, d\nu(y) = 0$$

    since $E - y$ is Lebesgue-null, hence μ-null, for all y.

35. The Fourier transform of a function f in complex $L^1(\lambda)$ is the function $\hat{f}$ on $\mathbb{R}$ defined by

$$\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx}\,dx.$$

(a) Prove that if f is in $L^1(\lambda)$ then $\hat{f}$ is continuous.
Proof: Since $|f(x)e^{-itx}| \le |f(x)|$, which is integrable, and $f(x)e^{-itx} \to f(x)e^{-it_0x}$ pointwise as $t \to t_0$, the dominated convergence theorem gives the desired result.

(b) For f in $L^1(\lambda)$ and y in $\mathbb{R}$, let T_yf be the y-translate of f: $(T_yf)(x) = f(x-y)$. Find the relation between $\hat{f}$ and $(T_yf)\hat{}$.
Solution: We have

$$(T_yf)\hat{}(t) = \int_{\mathbb{R}} f(x-y)e^{-itx}\,dx = \int_{\mathbb{R}} f(x)e^{-it(x+y)}\,dx = e^{-ity}\hat{f}(x).$$

(c) Prove that if f is in $L^1(\lambda)$ then $\lim_{|t|\to\infty} \hat{f}(t) = 0$. (Riemann-Lebesgue lemma)
Proof: By (b), $(T_{\pi/t}f)\hat{}(t) = e^{-i\pi}\hat{f}(t) = -\hat{f}(t)$. Hence

$$\hat{f}(t) = \frac{1}{2}(f - T_{\pi/t}f)\hat{}(t) = \frac{1}{2}\int_{\mathbb{R}} (f(x) - T_{\pi/t}f(x))e^{-itx}\,dx.$$

It follows that $|\hat{f}(t)| \le \|f - T_{\pi/t}f\|_1$, and the preceding norm tends to 0 as $|t| \to \infty$ by the continuity of translation in $L^1(\lambda)$.

(d) Prove that if f and g are in $L^1(\lambda)$ then $(f * g)\hat{} = \hat{f}\hat{g}$.
Proof: We use Fubini's theorem:

$$\begin{aligned}(f*g)\hat{}(t) &= \int_{\mathbb{R}} (f*g)(x)e^{-itx}\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)e^{-itx}\,dy\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)e^{-itx}\,dx\,dy = \int_{\mathbb{R}} f(x)e^{-ity}\hat{g}(t)\,dx \text{ (by (b))} \\ &= \hat{f}(t)\hat{g}(t).\end{aligned}$$

The use of Fubini's theorem is justified because, as shown when convolution was defined, the function $(x,y) \mapsto f(y)g(x-y)$ on $\mathbb{R}^2$ is integrable with respect to λ_2.

(e) Prove that if f is in $L^1(\lambda) \cap C^1(\mathbb{R})$ and f' is in $L^1(\lambda)$, then $(f')\hat{}(t) = it\hat{f}(t)$.
Proof 1: Let $\psi = \frac{1}{2}\chi_{(-1,1)}$, $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$ for $\epsilon > 0$. By a result from lecture (theorem 5.1), $\psi_\epsilon * f' \to f'$ in L^1-norm as $\epsilon \to 0$. It follows that $(\psi_\epsilon * f')\hat{} \to (f')\hat{}$ pointwise. We have

$$\begin{aligned}(\psi_\epsilon * f')(x) &= \int_{\mathbb{R}} \psi_\epsilon(y)f'(x-y)\,dy = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} f'(x-y)\,dy \\ &= \frac{1}{2\epsilon}(f(x+\epsilon) - f(x-\epsilon)) = \frac{1}{2\epsilon}(T_{-\epsilon}f(x) - T_\epsilon f(x)).\end{aligned}$$

By (b), $(\psi_\epsilon * f')\hat{}(t) = \frac{1}{2\epsilon}(e^{i\epsilon t} - e^{-i\epsilon t})\hat{f}(t) \to it\hat{f}(t)$ as $\epsilon \to 0$, and the desired equality follows.
Proof 2: Since f' is continuous and in $L^1(\lambda)$, $f(x) = f(0) + \int_{[0,x]} f'\,d\lambda$ for $x > 0$; therefore, $\lim_{x\to\infty} f(x) = f(0) + \int_{[0,\infty)} f'\,d\lambda$ exists. Similarly, $\lim_{x\to-\infty} f(x) = f(0) - \int_{(-\infty,0]} f'\,d\lambda$ exists. However, since $f \in L^1(\lambda)$, both these limits must be equal to 0.
Therefore, integration by parts gives

$$(f')\hat{}(t) = \int_{\mathbb{R}} f'(x)e^{-itx}\,dx = f(x)e^{-itx}\Big|_{x\to-\infty}^{x\to\infty} + it\int_{\mathbb{R}} f(x)e^{-itx}\,dx = it\hat{f}(t).$$