21. Suppose the sequence \((f_n)\) in \(L^1(\mu)\) converges almost everywhere to the function \(f\) in \(L^1(\mu)\), and that 
\[
\lim_{n \to \infty} ||f_n||_1 = ||f||_1.
\]
Prove that \(\lim_{n \to \infty} ||f - f_n||_1 = 0\).

**Proof 1:** For each \(n\), let \(E_n = \{|f_n| < 2|f|\}\), and let \(g_n = f_n \chi_{E_n}\). Then if \(f(x) \neq 0\) and \(f_n(x) \to f(x)\) as \(n \to \infty\), then \(x \in E_n\) for \(n\) sufficiently large, so \(g_n(x) \to f(x)\). Also, if \(f(x) = 0\), then \(g_n(x) = 0\) for each \(n\). Therefore, \(g_n \to f\) almost everywhere.

Also, \(|g_n| \leq 2|f|\) by construction, so by the dominated convergence theorem, 
\[
\int |g_n| \, d\mu \to \int |f| \, d\mu = ||f||_1 \quad \text{as} \quad n \to \infty.
\]
On the other hand, by assumption, 
\[
\int |f_n| \, d\mu \to \int |f| \, d\mu \quad \text{as} \quad n \to \infty.
\]
Therefore, 
\[
\int |f_n - g_n| \, d\mu \to 0 \quad \text{as} \quad n \to \infty.
\]
However, \(|f_n - g_n| = |f_n| - |f - f_n|\), so we see that \(||f_n - g_n||_1 \to 0\) as \(n \to \infty\).

We now have 
\[
||f - f_n||_1 \leq ||f - g_n||_1 + ||f_n - g_n||_1.
\]
However, \(|f - g_n| \leq 3|f|\), so by the dominated convergence theorem, \(||f - g_n||_1 \to 0\) as \(n \to \infty\). Thus, 
\[
||f - g_n||_1 + ||f_n - g_n||_1 \to 0 \quad \text{as} \quad n \to \infty,
\]
so \(\lim_{n \to \infty} ||f - f_n||_1 = 0\) also, as desired.

**Proof 2:** Consider the sequence of nonnegative functions \(g_n = |f| + |f_n| - |f - f_n|\). Then 
\[
\lim_{n \to \infty} g_n = \lim_{n \to \infty} (|f| + |f_n| - |f - f_n|) = 2|f|,
\]
while 
\[
\lim_{n \to \infty} \int g_n \, d\mu = \lim_{n \to \infty} \left( \int |f| \, d\mu + \int |f_n| \, d\mu - \int |f - f_n| \, d\mu \right)
\]
\[
= \int |f| \, d\mu + \lim_{n \to \infty} \int |f_n| \, d\mu - \limsup_{n \to \infty} \int |f - f_n| \, d\mu.
\]
Therefore, Fatou’s Lemma gives 
\[
2||f||_1 = \left( \liminf_{n \to \infty} g_n \right) \, d\mu \leq \liminf_{n \to \infty} \int g_n \, d\mu = 2||f||_1 - \limsup_{n \to \infty} \int |f - f_n| \, d\mu,
\]
which implies \(\limsup_{n \to \infty} ||f - f_n||_1 \leq 0\). Since obviously \(\liminf_{n \to \infty} ||f - f_n||_1 \geq 0\), this implies the desired result.

22. Let \((X, \mathcal{R}, \mu)\) and \((Y, \mathcal{S}, \nu)\) be \(\sigma\)-finite measure spaces. Let \(A\) be a set in the hereditary \(\sigma\)-ring generated by \(\mathcal{R}\) and \(B\) a set in the hereditary \(\sigma\)-ring generated by \(\mathcal{S}\). Prove that \((\mu \times \nu)^*(A \times B) = \mu^*(A) \nu^*(B)\).

**Proof:** If \(R \in \mathcal{R}\) and \(A \subseteq R\), then \(\mu^*(A) \leq \mu(R)\), and there is an \(R \in \mathcal{R}\) for which equality holds. (See the solution to problem 5 for the proof of this last statement.) Similarly, if \(S \in \mathcal{S}\) and \(B \subseteq S\), then \(\nu^*(B) \leq \nu(S)\), and there is an \(S\) for which equality holds. And if \(E \in \mathcal{R} \times \mathcal{S}\) and \(A \times B \subseteq E\), then \(\mu^*(A \times B) \leq \mu \times \nu(E)\), and there is an \(E\) for which equality holds. (We can use \(\mathcal{R} \times \mathcal{S}\) in the last statement, rather than the possibly larger \(\sigma\)-ring of \(\mu \times \nu\)-measurable sets, because every \(\mu \times \nu\)-measurable set differs from a set in \(\mathcal{R} \times \mathcal{S}\) by a null set.)

Take \(R \in \mathcal{R}\) such that \(A \subseteq R\) and \(\mu^*(A) = \mu(R)\), and \(S \in \mathcal{S}\) such that \(B \subseteq S\) and \(\nu^*(B) = \nu(S)\). Let \(E = R \times S\). Then \(A \times B \subseteq E\), and 
\[
(\mu \times \nu)^*(A \times B) \leq \mu \times \nu(R \times S) = \mu(R) \nu(S) = \mu^*(A) \nu^*(B).
\]
It only remains to prove the reverse inequality.

Let \(E \in \mathcal{R} \times \mathcal{S}\) with \(A \times B \subseteq E\), and \(\mu \times \nu(E) = (\mu \times \nu)^*(A \times B)\). Then the function \(y \to \mu(E^y)\) is \(\mathcal{S}\)-measurable. Thus, letting \(F = \{y \in Y : \mu(E^y) \geq \mu^*(A)\}\), we have \(F \in \mathcal{S}\), and \(B \subseteq F\) since \(A \subseteq E^y\) whenever \(y \in B\). Therefore, 
\[
(\mu \times \nu)^*(A \times B) = \mu \times \nu(E) \geq \int_F \mu(E^y) \, d\nu(y) \geq \mu^*(A) \nu(F) \geq \mu^*(A) \nu^*(B).
\]
23. (a) Prove that if \( f \) is in \( L^1(\mu) \) then \( \lim_{t \to \infty} t\Lambda_t(f) = 0 \).

**Proof:** Let \( E_t = \{|f| > t\} \), so that \( \mu(E_t) = \Lambda_t(f) \). Then \( t\Lambda_t(f) = \int_{E_t} t \, d\mu \leq \int_{E_t} |f| \, d\mu \). However, \( \int_{E_t} |f| \, d\mu \to 0 \) as \( t \to \infty \) by the dominated convergence theorem, since \( \bigcap E_t = \{|f| = \infty\} \) is a null set. Therefore, \( t\Lambda_t(f) \to 0 \) as \( t \to \infty \) also.

(b) Prove that if \( \mu \) is a finite measure, if \( f \) is a measurable function, and if there is a positive number \( \epsilon \) such that \( \Lambda_f(t) = O(1/t(\log t)^{1+\epsilon}) \) as \( t \to \infty \), then \( f \) is in \( L^1(\mu) \).

**Proof:** We have

\[
\int |f| \, d\mu = \int_0^\infty \Lambda_f(t) \, dt = \int_0^a \Lambda_f(t) \, dt + \int_a^\infty \Lambda_f(t) \, dt \\
\leq a\mu(X) + \int_a^\infty \Lambda_f(t) \, dt.
\]

By assumption, there are positive numbers \( a, c \) such that \( \Lambda_f(t) \leq c/t(\log t)^{1+\epsilon} \) for \( t \geq a \). (Without loss of generality assume \( a > 1 \).) Hence

\[
\int |f| \, d\mu \leq a\mu(X) + \int_a^\infty \frac{dt}{t(\log t)^{1+\epsilon}} = a\mu(X) + \int_{\log a}^\infty \frac{ds}{s^{1+\epsilon}} < \infty.
\]

24. Let \( f \) be in \( L^1(\lambda) \). Define the function \( g \) on \( \mathbb{R} \) by

\[ g(x) = \int_{\mathbb{R}} \frac{f(y)}{1 + x^2 + y^2} \, dy. \]

Prove that \( g \) is in \( L^1(\lambda) \), and that \( \|g\|_1 \leq \pi \|f\|_1 \).

**Proof:** It is easy to see that the function \( \mathbb{R}^2 \to \mathbb{R} \) defined by \((x, y) \mapsto f(y)\) is Lebesgue measurable. In addition, \( \frac{1}{1+x^2+y^2} \) is continuous and thus Borel measurable, so their product \( \frac{f(y)}{1+x^2+y^2} \) is Lebesgue measurable. Thus, the fiberwise integral \( g \) is measurable, and we have

\[
\|g\|_1 \leq \int_\mathbb{R} \left( \int_{\mathbb{R}} \frac{|f(y)|}{1 + x^2 + y^2} \, dy \right) \, dx.
\]

Since \( \int_\infty \frac{1}{1+x^2+y^2} \, dx = \frac{\pi}{\sqrt{4+y^2}} \leq \pi \), we get \( \|g\|_1 \leq \pi \int_\mathbb{R} |f(y)| \, dy = \pi \|f\|_1 \).