13. Let \((X, \mathcal{A}, \mu)\) be a finite measure space and let \(f\) be a nonnegative measurable function on \(X\). Prove that 
\[ f \text{ is integrable if and only if } \sum_{n=1}^{\infty} \mu(\{f > n\}) < \infty. \]

**Proof:** Let \(E_n = \{f > n\}\), and consider the function \(g = \sum_{n=1}^{\infty} \chi_{E_n}\). Then it is easy to see that \(g = n\) on \(E_n \setminus E_{n+1} = \{n < f \leq n + 1\}\), while \(g = 0\) if \(f \leq 1\); thus, we see that \(g \leq f \leq g + 1\). Therefore, 
\[ \int g \, d\mu \leq \int f \, d\mu \leq \int g \, d\mu + \mu(X). \]

Since \(\int g \, d\mu = \sum_{n=1}^{\infty} \mu(\{f > n\})\) by an easy application of the monotone convergence theorem, this implies that \(\int f \, d\mu\) is finite if and only if \(\sum_{n=1}^{\infty} \mu(\{f > n\})\) is.

14. For \(\alpha\) a real number, define the function \(f_\alpha\) on \(\mathbb{R}\) by \(f_\alpha(x) = |x|^{2\alpha}/(1 + x^2)\). Prove that \(f\) is Lebesgue integrable if and only if \(-\frac{1}{2} < \alpha < \frac{1}{2}\).

**Proof:** Since \(f\) is an even function, it will suffice to consider its integrability on \((0, \infty)\), and we can treat separately the two subintervals \((0, 1]\) and \([1, \infty)\).

For \(1 \leq x < \infty\) we have \(\frac{1}{2} x^{2\alpha - 2} \leq f_\alpha(x) \leq x^{2\alpha - 2}\), hence \(f_\alpha\) is Lebesgue integrable on \([1, \infty)\) if and only if the function \(x \mapsto x^{2\alpha - 2}\) is, which happens if and only if the improper Riemann integral converges since this function is nonnegative. By elementary calculus, this is equivalent to \(2\alpha - 2 < -1\), or \(\alpha < \frac{1}{2}\). Similarly, for \(0 < x \leq 1\) we have \(\frac{1}{2} x^{2\alpha} \leq f_\alpha(x) \leq x^{2\alpha}\), so \(f\) is integrable on \([0, 1]\) if and only if the function \(x \mapsto x^{2\alpha}\) is, which occurs if and only if \(\alpha > -\frac{1}{2}\).

(In fact, it’s easy to see using complex variables that \(\int f_\alpha \, d\lambda = \pi/\cos(\pi\alpha)\) for \(-\frac{1}{2} < \alpha < \frac{1}{2}\).)

15. Let \(f\) be a Lebesgue-integrable function on \(\mathbb{R}\). Prove that the series 
\[ \sum_{n=-\infty}^{\infty} f(x + n) \]
converges absolutely for almost every \(x\) in \(\mathbb{R}\).

**Proof:** Define \(g : \mathbb{R} \to [0, \infty]\) by 
\[ g(x) = \sum_{n=-\infty}^{\infty} |f(x + n)|; \]
we need to prove that \(g < \infty\) almost everywhere. However, since \(g\) is obviously periodic with period 1, it suffices to show that \(g(x) < \infty\) for almost every \(x \in [0, 1]\).

To see this, define \(g_m : \mathbb{R} \to [0, \infty]\) by 
\[ g_m(x) = \sum_{n=-m}^{m} |f(x + n)|. \]
Then by the monotone convergence theorem 
\[ \int_{[0,1]} g_m \, d\lambda \to \int_{[0,1]} g \, d\lambda \text{ as } m \to \infty. \]
But 
\[ \int_{[0,1]} g_m \, d\lambda = \sum_{n=-m}^{m} \int_{(0,1)} |f(x + n)| \, d\lambda(x) = \sum_{n=-m}^{m} \int_{[n,n+1]} |f(x)| \, d\lambda(x) = \int |f| \chi_{[-m,m+1]} \, d\lambda. \]

Another application of the monotone convergence theorem shows that 
\[ \int |f| \chi_{[-m,m+1]} \, d\lambda \to \int |f| \, d\lambda < \infty \]
as \(m \to \infty\), so \(\int_{[0,1]} g \, d\lambda < \infty\). This implies that \(\lambda([0,1] \cap \{g = \infty\}) = 0\), completing the proof.

16. Let \(f\) be a Lebesgue-integrable function on \(\mathbb{R}^N\). For \(r \geq 0\) let \(B_r = \{x \in \mathbb{R}^N : \|x\| \leq r\}\), and define the function \(g : [0, \infty) \to \mathbb{R}\) by 
\[ g(r) = \int_{B_r} f \, d\lambda_N \]
\((\lambda_N = \text{Lebesgue measure})\). Prove \(g\) is continuous.
Proof: Let $r_n \to r$ be any convergent sequence of nonnegative real numbers. Then clearly $\chi_{B_{r_n}} \to \chi_{B_r}$ everywhere except possibly on $\partial B_r = \{ x : \| x \| = r \}$. Therefore, assuming that $\partial B_r$ is Lebesgue null, we have $f\chi_{B_{r_n}} \to f\chi_{B_r}$ almost everywhere, and $|f\chi_{B_{r_n}}| \leq |f|$ with $f \in L^1(\lambda_N)$. Thus, by the dominated convergence theorem, $g(r_n) = \int f\chi_{B_{r_n}} \, d\lambda_N \to \int f\chi_{B_r} \, d\lambda_N = g(r)$, showing that $g$ is continuous.

To show $\partial B_r$ is Lebesgue null, we have for any $\epsilon > 0$ that $\partial B_r \subseteq B_r \setminus B_{r-\epsilon}$, so $\lambda_N(\partial B_r) \leq \lambda_N(B_r) - \lambda_N(B_{r-\epsilon}) = (r^N - (r-\epsilon)^N)\lambda_N(B_1)$. Since the last term approaches 0 as $\epsilon \to 0$, this shows that $\lambda_N(\partial B_r) = 0$. (Note: $\lambda_N(B_1) = \pi^{N/2}/\Gamma(\frac{N}{2} + 1)$.)