Math 202B Solutions

Assignment 13

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We first prove the following useful result, which will be used in several of the problems.

Proposition 1. *Let B be a Banach space.*

1. *If $(x_n)_{n=1}^\infty \subseteq B$ is a weakly convergent sequence, then $\sup_n \|x_n\| < \infty$.*
2. *If $(\phi_n)_{n=1}^\infty \subseteq B^*$ is a weak-* convergent sequence, then $\sup_n \|\phi_n\| < \infty$.*

Proof. We first prove 2. Let ϕ be the weak-* limit of ϕ_n; this means that for each $x \in B$, $\phi_n(x) \to \phi(x)$ as $n \to \infty$. Thus, $\sup_n |\phi_n(x)| < \infty$ for each $x \in B$. The uniform boundedness principle thus applies to give the desired result.

Now to prove 1, recall that we have a natural map $\Phi : B \to B^{**}$ defined by $\Phi_x(\psi) = \psi(x)$ for $\psi \in B^*$. Also recall that this map is an isometry. Thus if $x_n \to x$ weakly, then clearly $\Phi_{x_n} \to \Phi_x$ in the weak-* topology, so by the previous paragraph this means $\sup_n \|x_n\| = \sup_n \|\Phi_{x_n}\| < \infty$. $\square$

50. Let S be the subset of ℓ^2 consisting of the sequences x_{mn} $(m, n = 1, 2, \ldots, m \neq n)$ defined by

$$x_{mn}(k) = \begin{cases} 1, & k = m \\ m, & k = n \\ 0, & k \neq m, n. \end{cases}$$

Prove 0 is in the weak closure of S, but no sequence in S converges to 0.

Proof: Consider the sequences y_m defined by

$$y_m(k) = \begin{cases} 1, & k = m \\ 0, & k \neq m. \end{cases}$$

Let $\phi \in (\ell^2)^*$ be any linear functional, induced by some sequence $z \in \ell^2$. Then $\phi(y_m) = z(m) \to 0$ as $m \to \infty$, which shows that $y_m \to 0$ weakly.

Now $x_{mn} = y_m + m y_n$, so for any fixed m, $x_{mn} = y_m + m y_n \to y_m$ weakly as $n \to \infty$. Thus, $y_m \in \overline{S}^{wk}$. Since $y_m \to 0$ weakly, this implies $0 \in \overline{S}^{wk}$.

However, any sequence $(x_{m_i n_i})_{i=1}^\infty$ in S which converges weakly to 0 must be bounded in ℓ^2-norm by the above proposition, which implies $\{m_i\}$ is bounded. On the other hand, defining $z \in \ell^2$ by $z(n) = \frac{1}{n}$, we have $\langle z, x_{m_i n_i} \rangle = \frac{1}{m_i} + \frac{m_i}{n_i} \to 0$ as $i \to \infty$, which implies $m_i \to \infty$, a contradiction.

51. Prove that a weakly convergent sequence in ℓ^1 is norm convergent.

Proof 1: Suppose some sequence $(x_n)_{n=1}^\infty$ is weakly convergent but not norm convergent. By translation, we may assume $x_n \to 0$ weakly as $n \to \infty$, and by taking a subsequence we may assume $\|x_n\|_1$ is bounded away from zero. Finally, by scaling, we may assume $\|x_n\|_1 \geq 1$ for every n.

Since evaluation at any coordinate is a bounded functional on ℓ^1, we have $x_n \to 0$ coordinatewise. Now choose K_1 such that $\sum_{k=K_1+1}^\infty |x_1(k)| < \frac{1}{5}$. Then $\sum_{k=1}^{K_1} |x_n(k)| \to 0$ as $n \to \infty$, so we may find n_2 such that $\sum_{k=1}^{K_1} |x_{n_2}(k)| < \frac{1}{5}$. Choose K_2 such that $\sum_{k=K_2+1}^\infty |x_{n_2}(k)| < \frac{1}{5}$.

Continuing in this way, we get a subsequence $x_{n_1} = x_1, x_{n_2}, \ldots$ and a sequence of integers $K_0 = 0, K_1, K_2, \ldots$ such that $\sum_{k=1}^{K_{j-1}} |x_{n_j}| < \frac{1}{5}$ and $\sum_{k=K_j+1}^\infty |x_{n_j}| < \frac{1}{5}$ for each j. Now define

$$y(k) = \frac{|x_{n_j}(k)|}{x_{n_j}(k)} \text{ for } K_{j-1} < k \leq K_j.$$

(Let $y(k) = 1$ if $x_{n_j}(k) = 0$.) Then clearly $y \in \ell^\infty$ since $|y(k)| = 1$. Also,

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} x_{n_j}(k)y(k)\right| &\geq \left|\sum_{k=K_{j-1}+1}^{K_j} x_{n_j}(k)y(k)\right| - \left|\sum_{k=1}^{K_{j-1}} x_{n_j}(k)y(k)\right| - \left|\sum_{k=K_j+1}^{\infty} x_{n_j}(k)y(k)\right| \\
&\geq \sum_{k=K_{j-1}+1}^{K_j} |x_{n_j}(k)| - \sum_{k=1}^{K_{j-1}} |x_{n_j}(k)| - \sum_{k=K_j+1}^{\infty} |x_{n_j}(k)| \\
&= \sum_{k=1}^{\infty} |x_{n_j}(k)| - 2\sum_{k=1}^{K_{j-1}} |x_{n_j}(k)| - 2\sum_{k=K_j+1}^{\infty} |x_{n_j}(k)| > \|x_{n_j}\|_1 - \frac{4}{5} \geq \frac{1}{5}.
\end{aligned}$$

Therefore, $\langle x_n, y\rangle \not\to 0$ as $n \to \infty$, which is a contradiction since $\langle \cdot, y\rangle$ is a bounded linear functional on ℓ^1.

(Comment: The preceding proof is the prototype of the "gliding hump argument," much beloved by Banach spacemen and Banach spacewomen.)

Proof 2: Assume as before that $x_n \to 0$ weakly. Fix $\epsilon > 0$, and for each n define $S_n = \{y \in \ell^\infty : |\langle y, x_n\rangle| \leq \epsilon\}$. Then since $x_n \to 0$ weakly, we have

$$\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} S_n = \ell^\infty.$$

However, each set $\bigcap_{n=m}^\infty S_n$ is weak-* closed. Also, the weak-* topology on the closed unit ball A of ℓ^∞ is completely metrizable, for example using the norm $\|y\|' = \sum_{k=1}^\infty 2^{-k}|y(k)|$. Therefore, by the Baire category theorem, for some m, $\bigcap_{n=m}^\infty S_n$ contains a nonempty weak-* open set in A. By standard reasoning from replacing ϵ by $\epsilon/2$, we get that $\bigcap_{n=m}^\infty S_n$ also contains a weak-* open neighborhood of 0 in A.

In other words, there are $a_1, \ldots, a_r \in \ell^1$ such that $V(a_1, \ldots, a_r; 1) \cap A \subseteq \bigcap_{n=m}^\infty S_n$. Now choose N large enough that $\sum_{j=N+1}^\infty |a_i(j)| < 1$ for each i, and for each n define y_n such that $y_n(j) = 0$ for $j \leq N$, $y_n(j) = \frac{|x_n(j)|}{x_n(j)}$ for $j > N$. We then have $y_n \in V(a_1, \ldots, a_r; 1) \cap A \subseteq S_n$, so

$$\|x_n\| = \langle y_n, x_n\rangle + \sum_{j=1}^{N} |x_n(j)| \leq \epsilon + \sum_{n=1}^{N} |x_n(j)|.$$

Each term of the finite sum goes to zero as $n \to \infty$, which gives us $\limsup_{n\to\infty} \|x_n\| \leq \epsilon$. Since ϵ was arbitrary, this shows $\|x_n\| \to 0$.

52. Let X be a locally compact Hausdorff space. Prove that a sequence in $C_0(X)$ converges weakly if and only if it is norm bounded and converges pointwise to a function in $C_0(X)$.

    Proof: If $f_n \to f$ weakly, then $\{f_n\}$ is norm bounded by the above proposition, and $f_n(x) \to f(x)$ for each $x \in X$ since evaluation at x is a bounded linear functional on $C_0(X)$. Conversely, if $\{f_n\}$ is norm bounded, say $\|f_n\|_\infty \leq C$ for each n. Then if $f_n \to f$ pointwise, then for any finite Radon measure μ on X, $C \in L^1(\mu)$. Thus, $\int f_n \, d\mu \to \int f \, d\mu$ by the dominated convergence theorem. Since a general functional in $C_0(X)^*$ is induced by such a measure μ, $f_n \to f$ weakly.

53. Prove that a sequence in ℓ^∞ converges in the weak-star topology of ℓ^∞ (as the dual of ℓ^1) if and only if it is norm bounded and converges coordinatewise.

    Proof: If $x_n \to x$ in the weak-* topology, then $\{x_n\}$ is norm bounded by the above proposition, and $x_n(j) \to x(j)$ for each j since evaluation at j is induced by a sequence in ℓ^1. Conversely, if $\{x_n\}$ is norm bounded, say $\|x_n\|_\infty \leq C$ for each n. Then for any $y \in \ell^1$, $|x_n(j)y(j)| \leq C|y(j)|$, where the right hand side has finite sum. Thus, if $x_n \to x$ coordinatewise, then by the dominated convergence theorem, $\sum_{j=1}^\infty x_n(j)y(j) \to \sum_{j=1}^\infty x(j)y(j)$. This shows that $x_n \to x$ in the weak-* topology.

54. Let B_1 and B_2 be Banach spaces, and let $T : B_1 \to B_2$ be a linear transformation.

(a) Prove that, if T is continuous relative to the weak topologies of B_1 and B_2, then T is bounded.

Proof: We use the closed graph theorem. Thus, suppose $x_n \to x$ in norm in B_1, and $Tx_n \to y$ in norm in B_2. Then $x_n \to x$ weakly in B_1, so by the continuity condition $Tx_n \to Tx$ weakly in B_2. On the other hand, $Tx_n \to y$ weakly in B_2. Since the weak topology on B_2 is Hausdorff, this implies that $y = Tx$.

(b) Prove that, if T is continuous relative to the weak topology of B_1 and the norm topology of B_2, then T has finite rank (i.e., TB_1 has finite dimension).

Proof: By the continuity condition, we can find functionals $\phi_1, \ldots, \phi_n$ such that whenever $|\phi_i(x)| < 1$ for $1 \leq i \leq n$, then $\|Tx\| < 1$. By scaling, this implies that if $\phi_i(x) = 0$ for $1 \leq i \leq n$, then $Tx = 0$. In other words, the kernel of T contains the subspace $\bigcap_{i=1}^n \ker(\phi_i)$ of B_1, which has codimension at most n. This implies that TB_1 has dimension at most n.