Math 202B Solutions

Assignment 10

D. Sarason

36. Let B_1 and B_2 be Banach spaces, with the norm in each denoted by $\|\cdot\|$. Let p be a number in $[1, \infty]$.

(a) Prove that one gets a norm on $B_1 \oplus B_2$, the algebraic direct sum of B_1 and B_2, if one defines

$$\|x_1 \oplus x_2\| = \begin{cases} (\|x_1\|^p + \|x_2\|^p)^{1/p}, & 1 \le p < \infty \\ \max\{\|x_1\|, \|x_2\|\}, & p = \infty. \end{cases}$$

Proof: Without loss of generality, assume the scalar field is $\mathbb{R}$. We first prove the special case in which $B_1 = B_2 = \mathbb{R}$. To do this, let μ be counting measure on $\{1, 2\}$. Thus, for a function $f : \{1, 2\} \to \mathbb{R}$, $\int f \, d\mu = f(1) + f(2)$. This implies that for $p < \infty$,

$$\|f\|_p = (|f(1)|^p + |f(2)|^p)^{1/p} = \|f(1) \oplus f(2)\|.$$

This shows that $\|\cdot\|$ is indeed a norm in this case. In particular, writing out the triangle inequality gives

$$(|a_1 + b_1|^p + |a_2 + b_2|^p)^{1/p} \le (|a_1|^p + |a_2|^p)^{1/p} + (|b_1|^p + |b_2|^p)^{1/p}.$$

The case $p = \infty$ is similar.

Now for the general case, the only nontrivial part is the triangle inequality. For $p < \infty$, we have

$$\begin{aligned}\|(x_1 \oplus x_2) + (y_1 \oplus y_2)\| &= (\|x_1 + y_1\|^p + \|x_2 + y_2\|^p)^{1/p} \le ((\|x_1\| + \|y_1\|)^p + (\|x_2\| + \|y_2\|)^p)^{1/p} \\ &\le (\|x_1\|^p + \|x_2\|^p)^{1/p} + (\|y_1\|^p + \|y_2\|^p)^{1/p} = \|x_1 \oplus x_2\| + \|y_1 \oplus y_2\|,\end{aligned}$$

where the last inequality follows from the previous paragraph. Again, the case $p = \infty$ is similar.

(b) Let $B_1 \oplus_p B_2$ denote $B_1 \oplus B_2$ equipped with the preceding norm. Prove $B_1 \oplus_p B_2$ is complete.

Proof: Suppose $(x_{1n} \oplus x_{2n})_{n=1}^\infty$ is a Cauchy sequence in $B_1 \oplus_p B_2$. Then $\|x_{1m} - x_{1n}\| \le \|(x_{1m} \oplus x_{2m}) - (x_{1n} \oplus x_{2n})\| \to 0$ as $m, n \to \infty$, so $(x_{1n})_{n=1}^\infty$ is a Cauchy sequence in B_1, and similarly for $(x_{2n})_{n=1}^\infty$. Therefore, these sequences have limits $y_1 \in B_1$, $y_2 \in B_2$. Now for $p < \infty$, $\|(x_{1n} \oplus x_{2n}) - (y_1 \oplus y_2)\| = (\|x_{1n} - y_1\|^p + \|x_{2n} - y_2\|^p)^{1/p} \to 0$ as $n \to \infty$, showing that the original sequence has limit $y_1 \oplus y_2$ in $B_1 \oplus_p B_2$. The case $p = \infty$ is similar.

(c) Prove that the dual of $B_1 \oplus_p B_2$ equals $B_1^* \oplus_{p'} B_2^*$.

Proof: Let $\phi : B_1 \oplus_p B_2 \to \mathbb{R}$ be a functional, and define $\phi_1 : B_1 \to \mathbb{R}$ by $\phi_1(x_1) = \phi(x_1 \oplus 0)$, and $\phi_2 : B_2 \to \mathbb{R}$ by $\phi_2(x_2) = \phi(0 \oplus x_2)$. We then see that $\phi(x_1 \oplus x_2) = \phi_1(x_1) + \phi_2(x_2)$, and conversely for any ϕ_1, ϕ_2 we get a functional ϕ on $B_1 \oplus_p B_2$ defined by this formula. It is easy to see that ϕ is bounded if and only if ϕ_1 and ϕ_2 are. It remains to calculate $\|\phi\|$.

Note that Hölder's inequality, applied to the counting measure μ on $\{1, 2\}$, gives the inequality

$$|a_1 b_1| + |a_2 b_2| \le (|a_1|^p + |a_2|^p)^{1/p} (|b_1|^{p'} + |b_2|^{p'})^{1/p'}$$

for $1 < p < \infty$; for $p = 1$ or $p = \infty$ we get similar inequalities. Thus, for $x_1 \in B_1$, $x_2 \in B_2$, $1 < p < \infty$,

$$\begin{aligned}|\phi(x_1 \oplus x_2)| &= |\phi_1(x_1) + \phi_2(x_2)| \le \|\phi_1\| \, \|x_1\| + \|\phi_2\| \, \|x_2\| \\ &\le (\|\phi_1\|^{p'} + \|\phi_2\|^{p'})^{1/p'} (\|x_1\|^p + \|x_2\|^p)^{1/p}) = \|\phi_1 \oplus \phi_2\|_{p'} \|x_1 \oplus x_2\|_p.\end{aligned}$$

This shows that $\|\phi\| \le \|\phi_1 \oplus \phi_2\|_{p'}$. Similar proofs hold for $p = 1$ and for $p = \infty$.

Now for the opposite inequality, assume without loss of generality that ϕ_1, ϕ_2 are not both zero. For $1 < p < \infty$ set $m_1 = \|\phi_1\|^{p'/p}$ and $m_2 = \|\phi_2\|^{p'/p}$; and similarly, for $p = \infty$ set $m_1 = m_2 = 1$. If $\|x_i\| = m_i$ for $i = 1, 2$, we get equality in the application of Hölder's inequality above. Also, for any

$\epsilon > 0$ we can find $x_1 \in B_1$ and $x_2 \in B_2$ with $\|x_i\| = m_i$ and $\phi_i(x_i) > \|\phi_i\| m_i - \epsilon$ for $i = 1, 2$. Thus, letting $m = (m_1^p + m_2^p)^{1/p}$, or $m = \max\{|m_1|, |m_2|\}$ for $p = \infty$,

$$\phi(x_1 \oplus x_2) = \phi_1(x_1) + \phi_2(x_2) > \|\phi_1\| \, \|x_1\| + \|\phi_2\| \, \|x_2\| - 2\epsilon = \|\phi_1 \oplus \phi_2\|_{p'} \cdot m - 2\epsilon.$$

Since $\|x_1 \oplus x_2\|_p = m$, this shows that $\|\phi\| > \|\phi_1 \oplus \phi_2\|_{p'} - \frac{2}{m}\epsilon$. Since ϵ was arbitrary, this completes the proof in the case $1 < p \leq \infty$. On the other hand, for $p = 1$ it is obvious that $\|\phi_i\| \leq \|\phi\|$ for $i = 1, 2$. In other words, $\|\phi\| \geq \max\{\|\phi_1\|, \|\phi_2\|\}$, thus proving the desired inequality in this case also.

37. Prove that all norms on a finite-dimensional vector space B are equivalent: if $\|\cdot\|$ and $\|\cdot\|'$ are norms on B, then there are positive constants c_1 and c_2 such that

$$c_1\|x\| \leq \|x\|' \leq c_2\|x\|$$

for all x in B.

Proof: Since a complex vector space is also a real vector space, it suffices to consider the case of real scalars. It is straightforward to show that equivalence of norms, in the sense defined above, is an equivalence relation. Thus, it is sufficient to show that every norm on $B \simeq \mathbb{R}^N$ is equivalent to the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^N$.

To show this, let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^N$. Then for $x = x_1e_1 + \cdots + x_ne_n$,

$$\|x\| \leq |x_1| \, \|e_1\| + \cdots + |x_n| \, \|e_n\| \leq \|x\|_2 \sqrt{\|e_1\|^2 + \cdots + \|e_n\|^2}.$$

(The last inequality is just the Cauchy-Schwarz inequality.) Therefore, for $C = \sqrt{\|e_1\|^2 + \cdots + \|e_n\|^2}$, $\|x\| \leq C\|x\|_2$.

To get the other inequality, we observe that since $| \, \|x\| - \|y\| \, | \leq \|x - y\| \leq C\|x - y\|_2$, $\|\cdot\|$ is a continuous function with respect to the usual topology. Therefore, since $S^{N-1} = \{x : \|x\|_2 = 1\}$ is compact, $\|x\|$ achieves a minimum value $c > 0$ on S^{N-1}. Thus, if $x \neq 0$, we have $\frac{x}{\|x\|_2} \in S^{N-1}$, and

$$\|x\| = \|x\|_2 \left\| \frac{x}{\|x\|_2} \right\| \geq c\|x\|_2.$$

We are now done since the inequality is obvious for $x = 0$.

38. Prove that a finite dimensional subspace of a Banach space is closed.

Proof: It is sufficient to show a finite dimensional subspace A is complete with respect to the norm inherited from the Banach space. However, from the previous problem any norm on A is complete since it's equivalent to the usual norm on $\mathbb{R}^N$ or $\mathbb{C}^N$. (In particular, it's easy to see that equivalent norms give the same Cauchy sequences and convergent sequences.)

39. (a) Let $x_1, \ldots, x_n$ be linearly independent vectors in a Banach space B. Prove that there are functionals $\phi_1, \ldots, \phi_n$ in B^* such that $\phi_j(x_j) = 1$ for all j and $\phi_j(x_k) = 0$ for $j \neq k$.

Proof: For each j let A_j be the linear span of $\{x_k : k \neq j\}$. By the preceding problem A_j is closed. Since $x_j \notin A_j$, we have $d(x_j, A_j) > 0$. By the extension theorem, then, there is a $\phi_j \in B^*$ such that $\phi_j(x_j) = 1$ and $\phi_j|A_j = 0$. The functionals $\phi_1, \ldots, \phi_n$ have the required properties.

(b) Let A be a finite-dimensional subspace of a Banach space B. Prove that there is a closed subspace A' of B such that $A \cap A' = \{0\}$ and $A + A' = B$.

Proof: Let $\{x_1, \ldots, x_n\}$ be a basis for A, let $\phi_j \in B^*$ be as in the previous part, and define $A' = \bigcap_{j=1}^n \ker(\phi_j)$. Then if $x \in A \cap A'$, there are scalars $a_1, \ldots, a_n$ such that $x = a_1x_1 + \cdots + a_nx_n$ since $x \in A$. Applying ϕ_j, we get $a_j = \phi_j(x) = 0$ for each j since $x \in A'$. Thus, $x = 0$, showing that $A \cap A' = \{0\}$.

Now let $x \in B$, and let $y = \sum_{j=1}^n \phi_j(x)x_j \in A$. Then $\phi_j(x - y) = 0$ for all j, so that $x - y \in A'$. Hence $A + A' = B$.

40. Consider the Banach space ℓ^∞ (real scalars). Let $T : \ell^\infty \to \ell^\infty$ be the shift operator on ℓ^∞, the map that sends $x = (x_1, x_2, \ldots)$ in ℓ^∞ to $Tx = (0, x_1, x_2, \ldots)$. Let $Y = \{x - Tx : x \in \ell^\infty\}$ and let e be the sequence $(1, 1, \ldots)$.

(a) Prove $\text{dist}(e, Y) = 1$.

Proof: Let $x = (x_1, x_2, \dots) \in \ell^\infty$. Then since x is bounded, for each $\epsilon > 0$ there is some n such that $x_n - x_{n-1} < \epsilon$. This implies that $(e - (x - Tx))_n = 1 - (x_n - x_{n-1}) > 1 - \epsilon$, so $\|e - (x - Tx)\|_\infty > 1 - \epsilon$. Since ϵ was arbitrary, this shows $\text{dist}(e, x - Tx) \geq 1$ for each $x \in \ell^\infty$, so $\text{dist}(e, Y) \geq 1$. However, since $\text{dist}(e, 0) = 1$, we get $\text{dist}(e, Y) \leq 1$ also.

(b) Prove there is a ϕ in $(\ell^\infty)^*$ such that $\phi(e) = 1$, $\|\phi\| = 1$, and $\phi = 0$ on Y.

Proof: This follows from the previous part by a corollary to the Hahn-Banach extension theorem.

(c) Prove ϕ is translation invariant: $\phi(Tx) = \phi(x)$ for all x.

Proof: This is a corollary of the fact that $\phi = 0$ on Y, so $\phi(x - Tx) = 0$ for each $x \in \ell^\infty$.

(d) Prove that $\liminf_{n\to\infty} x_n \leq \phi(x) \leq \limsup_{n\to\infty} x_n$ for all x. (In particular, $\phi(x) = \lim_{n\to\infty} x_n$ if x converges.) (Banach. Such a functional ϕ is called a Banach limit.)

Proof: We first show that $\inf\{x_n\} \leq \phi(x) \leq \sup\{x_n\}$. To do this, let $m = \inf\{x_n\}$, $M = \sup\{x_n\}$. Since $\|\phi\| = 1$, we get

$$\left|\phi(x) - \frac{M+m}{2}\right| = \left|\phi\left(x - \frac{M+m}{2}e\right)\right| \leq \left\|x - \frac{M+m}{2}e\right\|_\infty = \frac{M-m}{2}.$$

Thus, $m \leq \phi(x) \leq M$.

Now let $S : \ell^\infty \to \ell^\infty$ be the reverse shift operator defined by $S(x) = (x_2, x_3, \dots)$. Then since $TS(x) = x - x_1(e - Te)$, the translation invariance of ϕ gives that $\phi(Sx) = \phi(x)$. Now applying the previous paragraph to $S^m x$ gives

$$\inf\{x_n : n > m\} \leq \phi(x) \leq \sup\{x_n : n > m\}$$

for every m. Taking the limit as $m \to \infty$ gives the desired result.