32. Let $\mathcal{A}$ be a $\sigma$-algebra on a set $X$.

(a) Prove that if $\mu$ is a positive $\sigma$-finite measure on $\mathcal{A}$, then there is a finite measure on $\mathcal{A}$ that is mutually absolutely continuous with respect to $\mu$.

(b) Let $\mu_1, \mu_2, \ldots$ be positive $\sigma$-finite measures on $\mathcal{A}$. Prove that there is a finite measure $\nu$ on $\mathcal{A}$ such that $\mu_n \ll \nu$ for all $n$.

33. Let $\mathcal{A}$ be a $\sigma$-algebra on a set $X$. Let $\mu$ and $\nu$ be positive measures in $M(\mathcal{A})$ such that $\|\mu - \nu\| = \|\mu\| + \|\nu\|$. Prove that $\mu \perp \nu$.

34. Let $\mu$ and $\nu$ be measures in $M(\mathbb{R}^N)$ such that $\mu \ll \lambda_N$. Prove that $\mu * \nu \ll \lambda_N$.

35. The Fourier transform of a function $f$ in complex $L^1(\lambda)$ is the function $\hat{f}$ on $\mathbb{R}$ defined by

$$\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx}dx.$$ 

(a) Prove that if $f$ is in $L^1(\lambda)$ then $\hat{f}$ is continuous.

(b) For $f$ in $L^1(\lambda)$ and $y$ in $\mathbb{R}$, let $T_yf$ be the $y$-translate of $f$ : $(T_yf)(x) = f(x - y)$. Find the relation between $\hat{f}$ and $(T_yf)^\wedge$.

(c) Prove that if $f$ is in $L^1(\lambda)$ then $\lim_{|t| \to \infty} \hat{f}(t) = 0$. (Riemann–Lebesgue lemma) (Suggestion: Along with $f$ consider $T_{\pi/t}f$.)

(d) Prove that if $f$ and $g$ are in $L^1(\lambda)$ then $(f * g)^\wedge = \hat{f} \hat{g}$. 

(e) Prove that if $f$ is in $L^1(\lambda) \cap C^1(\mathbb{R})$ and $f'$ is in $L^1(\lambda)$, then $(f')^\wedge(t) = it\hat{f}(t)$. (Suggestion: Consider $\psi_\epsilon * f'$, where $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon}\right)$, $\psi = \frac{1}{2}\chi_{(-1,1)}$.)