HOMEWORK ASSIGNMENT 8

Due in class on Friday, March 19.

29. Let $\mu$ be a signed measure on a $\sigma$-ring $R$. Suppose $\mu_1$ and $\mu_2$ are nonnegative measures on $R$, at least one of which is finite, such that $\mu = \mu_1 - \mu_2$. Prove that $\mu_+ \leq \mu_1$ and $\mu_- \leq \mu_2$ (i.e., $\mu_+(E) \leq \mu_1(E)$ and $\mu_-(E) \leq \mu_2(E)$ for every set $E$ in $R$).

30. Let $A$ be a $\sigma$-algebra on a set $X$. Let $M(A)$ be the space of finite signed measure on $A$, made into a real vector space under the operations

\[
(c\mu)(E) = c\mu(E) \quad (\mu \in M(A), c \in \mathbb{R}, E \in A)
\]

\[
(\mu_1 + \mu_2)(E) = \mu_1(E) + \mu_2(E) \quad (\mu_1, \mu_2 \in M(A), E \in A).
\]

For $\mu$ in $M(A)$ let $\|\mu\| = |\mu|(X)$. Prove that $\|\cdot\|$ is a norm on $M(A)$, and that $M(A)$ is complete under this norm.

31. Let $M(\mathbb{R}^N)$ be the space of finite signed measures on $B(\mathbb{R}^N)$, the Borel $\sigma$-algebra on $\mathbb{R}^N$, made into a normed space as in Problem 30. For $\mu$ and $\nu$ in $M(\mathbb{R}^N)$, define $\mu \ast \nu : B(\mathbb{R}^N) \to \mathbb{R}$ by

\[
\mu \ast \nu(E) = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \chi_E(x + y)d\nu(y) \right) d\mu(x),
\]

in other words, $\mu \ast \nu(E) = (\mu \times \nu)(A_E)$, where $A_E = \{(x, y) \in \mathbb{R}^{2N} : x + y \in E\}$.

(a) Prove that $\mu \ast \nu$ is in $M(\mathbb{R}^N)$.

(b) Prove that if $f$ is a bounded Borel-measurable function on $\mathbb{R}^N$, then

\[
\int_{\mathbb{R}^N} f d(\mu \ast \nu) = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} f(x + y)d\nu(y) \right) d\mu(x).
\]

(c) Prove that $(\mu \ast \nu) \ast \xi = \mu \ast (\nu \ast \xi)$.

(d) Prove that $\|\mu \ast \nu\| \leq \|\mu\|\|\nu\|$.

(e) For $f$ a Lebesgue-integrable function on $\mathbb{R}^N$, define the measure $\mu_f$ in $M(\mathbb{R}^N)$ by $\mu_f(E) = \int_E f \ d\lambda_N$. Prove that $\mu_f \ast \mu_g = \mu_{f \ast g}$ for any two Lebesgue-integrable functions $f$ and $g$. (Thus, convolution of measures generalizes convolution in $L^1(\lambda_N)$.)