HOMEWORK ASSIGNMENT 4

Due in class on Friday, February 20.

13. Let \((X, \mathcal{A}, \mu)\) be a finite measure space and let \(f\) be a nonnegative measurable function on \(X\). Prove that \(f\) is integrable if and only if
\[
\sum_{n=1}^{\infty} \mu(\{f > n\}) < \infty.
\]

14. For \(\alpha\) a real number, define the function \(f_\alpha\) on \(\mathbb{R}\) by \(f_\alpha(x) = |x|^{2\alpha}/(1 + x^2)\). Prove that \(f\) is Lebesgue integrable if and only if \(-\frac{1}{2} < \alpha < \frac{1}{2}\).

15. Let \(f\) be a Lebesgue-integrable function on \(\mathbb{R}\). Prove that the series
\[
\sum_{n=-\infty}^{\infty} f(x + n)
\]
converges absolutely for almost every \(x\) in \(\mathbb{R}\).

16. Let \(f\) be a Lebesgue-integrable function on \(\mathbb{R}^N\). For \(r \geq 0\) let \(B_r = \{x \in \mathbb{R}^N : \|x\| \leq r\}\), and define the function \(g : [0, \infty) \to \mathbb{R}\) by
\[
g(r) = \int_{B_r} f \, d\lambda_N
\]
\((\lambda_N = \text{Lebesgue measure})\). Prove \(g\) is continuous.