HOMEWORK ASSIGNMENT 1

Due in class on Friday, January 30.

1. Let $\mathcal{R}$ be a ring on a set $X$, let $\mathcal{R}'$ be the family of complements of the sets in $\mathcal{R}$, and let $A = \mathcal{R} \cup \mathcal{R}'$. Prove that $A$ is an algebra, and is a $\sigma$-algebra if $\mathcal{R}$ is a $\sigma$-ring.

2. Let $\mathcal{L}$ be a lattice of sets, that is, a family of sets that contains $\emptyset$ and is closed under finite unions and finite intersections. Prove that the family of relative complements of the sets in $\mathcal{L}$ is a semiring.

3. Let $X$ be a complete metric space, and let $\mathcal{A}$ be the family of subsets of $X$ that are either meager or residual. For $A$ in $\mathcal{A}$ define

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is meager} \\ 1 & \text{if } A \text{ is residual.} \end{cases}$$

Prove that $\mathcal{A}$ is a $\sigma$-algebra and that $\mu$ is a measure.

4. Let $X = \{0, 1\}^\mathbb{N}$, the set of all sequences of 0’s and 1’s (aka the coin-tossing space), regarded as a topological space with the product topology (each coordinate space $\{0, 1\}$ having the discrete topology). For $n$ in $\mathbb{N}$ let $P_n$ denote the $n$-th coordinate projection on $X$, the function that maps a sequence in $X$ to its $n$-th term. Recall that the subbasic open sets in $X$ are the sets $P_n^{-1}(\varepsilon)$ ($n \in \mathbb{N}, \varepsilon \in \{0, 1\}$), and the basic open sets are the finite intersections of subbasic open sets.

(a) Prove the Borel $\sigma$-algebra on $X$ is the $\sigma$-algebra generated by the basic open sets.

(b) Prove the basic open sets, together with $\emptyset$, form a semiring.

(Suggestion: The following notation may be helpful. For $S$ a finite subset of $\mathbb{N}$ and $f : S \to \{0, 1\}$, let

$$U(S, f) = \{x = (\varepsilon_m)_{m=1}^\infty \in X : \varepsilon_n = f(n) \text{ for } n \in S\}.$$ 

This is a typical basic open set.)