Math 202B **Spring 2004**
Sarason

HOMEWORK ASSIGNMENT 1

Due in class on Friday, January 30.

1. Let $\mathcal{R}$ be a ring on a set X, let $\mathcal{R}'$ be the family of complements of the sets in $\mathcal{R}$, and let $\mathcal{A} = \mathcal{R} \cup \mathcal{R}'$. Prove that $\mathcal{A}$ is an algebra, and is a σ-algebra if $\mathcal{R}$ is a σ-ring.

2. Let $\mathcal{L}$ be a lattice of sets, that is, a family of sets that contains $\emptyset$ and is closed under finite unions and finite intersections. Prove that the family of relative complements of the sets in $\mathcal{L}$ is a semiring.

3. Let X be a complete metric space, and let $\mathcal{A}$ be the family of subsets of X that are either meager or residual. For A in $\mathcal{A}$ define

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is meager} \\ 1 & \text{if } A \text{ is residual.} \end{cases}$$

   Prove that $\mathcal{A}$ is a σ-algebra and that μ is a measure.

4. Let $X = \{0,1\}^{\mathbb{N}}$, the set of all sequences of 0's and 1's (aka the coin-tossing space), regarded as a topological space with the product topology (each coordinate space $\{0,1\}$ having the discrete topology). For n in $\mathbb{N}$ let P_n denote the n-th coordinate projection on X, the function that maps a sequence in X to its n-th term. Recall that the subbasic open sets in X are the sets $P_n^{-1}(\varepsilon)$ $(n \in \mathbb{N}, \varepsilon \in \{0,1\})$, and the basic open sets are the finite intersections of subbasic open sets.

   (a) Prove the Borel σ-algebra on X is the σ-algebra generated by the basic open sets.

   (b) Prove the basic open sets, together with ϕ, form a semiring.
   (Suggestion: The following notation may be helpful. For S a finite subset of $\mathbb{N}$ and $f : S \to \{0,1\}$, let

$$U(S,f) = \{x = (\varepsilon_m)_1^\infty \in X : \varepsilon_n = f(n) \text{ for } n \in S\}.$$

   This is a typical basic open set.)