SUGGESTED EXERCISES

Pages 260–261, Exercises 33.3(a), 33.4, 33.7, 33.8, 33.9, 33.12, 33.14, plus Exercises 1–12 below.

1. Let $C_1$ and $C_2$ be connected subsets of a metric space such that $C_1 \cap C_2 = \emptyset$ and $\bar{C}_1 \cap C_2 \neq \emptyset$. Prove $C_1 \cup C_2$ is connected.

2. Let $A$ and $B$ be connected subsets of $\mathbb{R}^k$. Prove the set
   \[ A + B = \{a + b : a \in A, \ b \in B\} \]
   is connected.

3. Prove the set
   \[ C = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\} \]
   is connected.

4. Prove that, for $k \geq 2$, the set
   \[ S^{k-1} = \{x \in \mathbb{R}^k : \|x\| = 1\}, \]
   the unit sphere in $\mathbb{R}^k$, is connected. (Suggestion: Use induction on $k$.)

5. Prove there is no one-to-one continuous function of $\mathbb{R}^2$ onto $\mathbb{R}$.

6. Let $I \subset \mathbb{R}$ be an open interval, and let the function $f : I \rightarrow \mathbb{R}$ be differentiable to second order, with $f''$ continuous. Prove that, for $x$ in $I$,
   \[ f''(x) = \lim_{\delta \to 0} \frac{f(x + \delta) + f(x - \delta) - 2f(x)}{\delta^2}. \]

7. Let $I \subset \mathbb{R}$ be an open interval. Let $f : I \rightarrow \mathbb{R}$ satisfy
   \[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} > 0 \]
   for all $x$ in $I$. Prove $f$ is increasing.

8. Let $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable and let $g : [a, b] \rightarrow \mathbb{R}$ be bounded. Prove $U(f + g) = U(f) + U(g)$ and $L(f + g) = L(f) + L(g)$.

9. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Assume $f$ is Riemann integrable on $[c, b]$ for every $c$ in $(a, b)$. Prove $f$ is Riemann integrable on $[a, b]$.

10. Prove that the characteristic function of the Cantor set is Riemann integrable and that its integral is 0.

11. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Prove that for $f$ to be Riemann integrable it is necessary and sufficient that, for every $\varepsilon > 0$, there are continuous functions $g : [a, b] \rightarrow \mathbb{R}$ and $h : [a, b] \rightarrow \mathbb{R}$ such that $g \leq f \leq h$ and $\int_a^b h - \int_a^b g < \varepsilon$. 
12. Let \( f : [a, b] \to \mathbb{R} \) be differentiable, with \(|f'| \leq M\). Fix a positive integer \( n \), and let \( t_k = a + \frac{k}{n}(b - a) \) for \( k = 0, 1, \ldots, n \), so that \((t_0, t_1, \ldots, t_n)\) is the partition of \([a, b]\) for which each subinterval has length \( \frac{b - a}{n} \). For each \( k \) let \( x_k = \frac{t_{k-1} + t_k}{2} \) be the midpoint of \([t_{k-1}, t_k]\), and let \( R_n = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) \). Prove that the absolute value of the difference between the Riemann sum \( R_n \) and \( \int_{a}^{b} f \) is bounded by \( \frac{M(b - a)^2}{4n} \).