HOMEWORK ASSIGNMENT 6

Due in class on Wednesday, October 15.

Exercise L is optional.

K. Prove that every point of the Cantor set is a limit point of the Cantor set.

L. Let the series \( \sum_{k=1}^{\infty} a_k \) converge but not converge absolutely.
   (a) Prove the series has a rearrangement that diverges.
   (b) Let \( c \) be a real number. Prove the series has a rearrangement that converges to \( c \).

M. Prove as follows that if \( \sum_{k=1}^{\infty} a_k \) is a convergent series and \( (\beta_k)_{1}^{\infty} \) is a nonincreasing sequence with limit 0, then \( \sum_{k=1}^{\infty} a_k \beta_k \) converges.

   **Step 1.** Let \( \alpha_n = \sum_{k=1}^{n} a_k \) \( (n = 1, 2, \ldots) \) and \( b_k = \beta_k - \beta_{k-1} \) \( (k = 2, 3, \ldots) \). Show that, for \( 1 < m < n \),
   \[
   \sum_{k=m}^{n} a_k \beta_k = \alpha_n \beta_n - \alpha_{m-1} \beta_m - \sum_{k=m}^{n-1} a_k b_{k+1}.
   \]

   **Step 2.** Use the preceding identity to show that the partial sums of \( \sum_{k=1}^{\infty} a_k \beta_k \) form a Cauchy sequence.
   (Comment: The preceding technique is called summation by parts. Note the analogy with integration by parts.)

N. Prove the parallelogram equality for vectors \( x \) and \( y \) in \( \mathbb{R}^k \):
   \[
   \| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2).
   \]

O. Prove that \( \ell^\infty \) is complete.