MIDTERM EXAMINATION 1 – SOLUTIONS

1. For which values of $a$ and $b$ does the straight line $y = ax + b$ give the best least-squares approximation to the data points $(0, 0)$, $(1, 2)$, $(2, 3)$?

Solution. The problem is to minimize the function

$$E(a, b) = b^2 + (a + b - 2)^2 + (2a + b - 3)^2.$$ 

We look for critical points of $E$. We have

$$\frac{\partial E}{\partial a} = 2(a + b - 2) + 4(2a + b - 3) = 10a + 6b - 16,$$
$$\frac{\partial E}{\partial b} = 2b + 2(a + b - 2) + 2(2a + b - 3) = 6a + 6b - 10.$$

The critical points $(a, b)$ are the solutions of the pair of equations

$$10a + 6b = 16, \quad 6a + 6b = 10$$

Subtracting the second equation from the first, we obtain $4a = 6$, or $a = 3/2$. Replacing $a$ by $3/2$ in the second equation, we get $9 + 6b = 10$, or $b = 1/6$.

Conclusion. There is only one critical point, $(3/2, 1/6)$. The desired straight line is $y = \frac{3}{2} x + \frac{1}{6}$.

2. For the function $f(x, y) = x^3 + y^3 - 12x - 27y$:

(a) Determine the critical points.
(b) Determine which critical points, if any, are saddle points.

Solution. (a) We have

$$\frac{\partial f}{\partial x} = 3x^2 - 12, \quad \frac{\partial f}{\partial y} = 3y^2 - 27.$$ 

Setting $\frac{\partial f}{\partial x} = 0$, we get the solutions $x = \pm 2$. Setting $\frac{\partial f}{\partial y} = 0$, we get the solutions $y = \pm 3$.

Conclusion. There are four critical points: $(2, 3)$, $(2, -3)$, $(-2, 3)$, $(-2, -3)$.

(b) We have

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$ 

Thus

$$D_f = \left(\frac{\partial^2 f}{\partial x^2}\right) \left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 36xy.$$ 

A critical point $(a, b)$ of $f$ is a saddle point if and only if $D_f(a, b) < 0$, which in our case happens if and only if $ab < 0$.

Conclusion. The critical points $(2, -3)$ and $(-2, 3)$ are saddle points of $f$. The other two critical points are not saddle points. (For the record, $(2, 3)$ is a relative minimum and $(-2, -3)$ is a relative maximum.)
3. Evaluate the integral \( I = \int \int_R (x^2 - y) \, dx \, dy \), where the region \( R \) is defined by the inequalities \( 0 \leq y \leq 1, \, x^2 \leq y \).

**Solution 1.** We integrate with respect to \( x \) first:

\[
I = \int_{0}^{1} \left[ \int_{\sqrt[3]{y}}^{\sqrt[3]{y}} (x^2 - y) \, dx \right] dy = \int_{0}^{1} \left[ \left( \frac{x^3}{3} - xy \right) \bigg|_{\sqrt[3]{y}}^{\sqrt[3]{y}} \right] dy
\]

\[
= \int_{0}^{1} \left( \frac{y^{3/2}}{3} - \frac{y^{3/2}}{3} + \frac{y^{3/2}}{3} - \frac{y^{3/2}}{3} \right) dy = \int_{0}^{1} -\frac{4}{3} \frac{y^{3/2}}{3} dy
\]

\[
= -\frac{4}{3} \left( \frac{2}{5} y^{5/2} \right) \bigg|_{0}^{1} = -\frac{8}{15} \text{ (Answer)}
\]

**Solution 2.** We integrate with respect to \( y \) first:

\[
I = \int_{-1}^{1} \left[ \int_{-\sqrt{y}}^{\sqrt{y}} (x^2 - y) \, dx \right] dy = \int_{-1}^{1} \left[ \left( x^2 y - \frac{y^2}{2} \right) \bigg|_{x^2}^{1} \right] dx
\]

\[
= \int_{-1}^{1} \left( x^2 - \frac{1}{2} - x^4 + \frac{x^4}{2} \right) dx
\]

\[
= \int_{-1}^{1} \left( x^2 - \frac{1}{2} - \frac{x^4}{2} \right) dx
\]

\[
= \left( \frac{x^3}{3} - \frac{x^2}{2} - \frac{x^5}{10} \right) \bigg|_{-1}^{1}
\]

\[
= \frac{1}{3} - \frac{1}{2} - \frac{1}{10} + \frac{1}{3} - \frac{1}{2} - \frac{1}{10} = \frac{10 - 15 - 3 + 10 - 15 - 3}{30} = -\frac{16}{30} = -\frac{8}{15} \text{ (Answer)}
\]

4. According to U.S. postal regulations, a rectangular package whose three side lengths measure \( x \) inches, \( y \) inches, \( z \) inches, must satisfy \( 2x + 2y + z \leq 84 \). Which values of \( x, y, z \) give the dimensions of a package whose diagonal has maximum length? (The length of the diagonal equals \( \sqrt{x^2 + y^2 + z^2} \).)

**Solution.** We use the Lagrange method, rephrasing the problem as: Maximize the function \( f(x, y, z) = x^2 + y^2 + z^2 \) under the constant \( g(x, y) = 84 - 2x - 2y - z = 0 \). We introduce the auxiliary variable \( \lambda \) and the auxiliary function

\[
F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z) = x^2 + y^2 + z^2 + \lambda(84 - 2x - 2y - z).
\]

We look for critical points of \( F \). We have

\[
\frac{\partial F}{\partial x} = 2x - 2\lambda, \quad \frac{\partial F}{\partial y} = 2y - 2\lambda, \quad \frac{\partial F}{\partial z} = 2z - \lambda.
\]
Setting \( \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0 \) and solving the resulting equations for \( \lambda \), we find that \( \lambda = x \), \( \lambda = y \), \( \lambda = 2z \), which tells us that \( x = y = 2z \). Substituting \( 2z \) for \( x \) and \( y \) in the constraint \( 2x + 2y + z = 84 \) gives us

\[
4z + 4z + z = 9z = 84,
\]

and accordingly \( z = 84/9 = 9\frac{1}{3} \). Thus \( x = 2z = 18\frac{2}{3} \), \( y = 2z = 18\frac{2}{3} \). We see that \( F \) has one critical point, whose \( x, y, z \) coordinates are \( 18\frac{2}{3}, 18\frac{2}{3}, 9\frac{1}{3} \).

**Conclusion.** The package with longest diagonal measures \( 18\frac{2}{3} \) inches by \( 18\frac{2}{3} \) inches by \( 9\frac{1}{3} \) inches. (The diagonal of this package turns out to be 28 inches long.)