Math 16B – S06 – Supplementary Notes 4
The Derivatives of $\sin t$ and $\cos t$

One can derive the formulas

(1) \[ \frac{d}{dt}(\sin t) = \cos t, \quad \frac{d}{dt}(\cos t) = -\sin t \]

starting from the relations

(2) \[ \lim_{t \to 0} \frac{\sin t}{t} = 1 \]

(3) \[ \lim_{t \to 0} \frac{\cos t - 1}{t} = 0. \]

Note that (2) and (3) just say that (1) holds at the origin (since $\cos 0 = 1$ and $\sin 0 = 0$).

Taking (2) and (3) temporarily for granted, let’s derive (1). By definition of the derivative,

\[ \frac{d}{dt}(\sin t) = \lim_{h \to 0} \frac{\sin(t + h) - \sin t}{h}. \]

We use the addition formula $\sin(t + h) = \sin t \cos h + \cos t \sin h$ to rewrite this as

\[ \frac{d}{dt}(\sin t) = \lim_{h \to 0} \left[ \sin t \left( \frac{\cos h - 1}{h} \right) + \cos t \left( \frac{\sin h}{h} \right) \right]. \]

By (2) and (3) the limit on the right side equals $\cos t$, which establishes the first formula in (1).

The second formula in (1) can be deduced from the first one by means of the identities

\[ \cos t = \sin \left( t + \frac{\pi}{2} \right), \quad \sin t = -\cos \left( t + \frac{\pi}{2} \right) \]

and the chain rule. We have

\[ \frac{d}{dt}(\cos t) = \frac{d}{dt} \left( \sin \left( t + \frac{\pi}{2} \right) \right) = \cos \left( t + \frac{\pi}{2} \right) \frac{d}{dt} \left( t + \frac{\pi}{2} \right) = \cos \left( t + \frac{\pi}{2} \right) = -\sin t. \]

So, to establish (1), it only remains to establish (2) and (3). Once (2) is known (3) follows easily. In fact,

\[ \frac{\cos t - 1}{t} = \frac{(\cos t - 1)(\cos t + 1)}{t(\cos t + 1)} = \frac{\cos^2 t - 1}{t(\cos t + 1)} = \frac{-\sin^2 t}{t(\cos t + 1)} = -\sin \left( \frac{\sin t}{t} \right) \left( \frac{1}{\cos t + 1} \right). \]

As $t$ tends to 0, the first factor on the right side tends to 0 (since $\sin 0 = 0$) and the last factor tends to $\frac{1}{2}$ (since $\cos 0 = 1$). By (2), the middle factor tends to 1, so the product tends to $(0)(1) \left( \frac{1}{2} \right) = 0$, which gives (3).

The relation (2) is thus the basic one. We’ll derive it using some simple geometry. Let $t$ be a small positive angle. (Since $\frac{\sin t}{t}$ is an even function of $t$, it suffices to establish (2) as $t$ tends to
0 through positive values.) From the point \( A = (\cos t, \sin t) \) on the unit circle we construct the tangent line to the circle, and we let \( B \) denote the point where the tangent line intersects the \( x \)-axis (see Figure 4.1). The origin will be denoted by \( O \).

The distance of the point \( A \) from the \( x \)-axis is \( \sin t \), which is less than the length of the arc of the unit circle subtended by the angle \( t \). The preceding arc has length \( t \) (by the definition of radians), so we have the inequality \( \sin t < t \), which we can write as \( \frac{\sin t}{t} < 1 \).

To obtain a lower bound for \( \frac{\sin t}{t} \) we consider the right triangle \( OAB \). The side adjacent to the angle \( t \) has length 1, so the side opposite the angle \( t \) has length \( \tan t \). The area of the triangle is therefore \( \frac{1}{2}(1)(\tan t) = \frac{\tan t}{2} \). The triangle contains the sector of the unit circle cut off by the angle \( t \), so its area is larger than the area of the sector. The area of the sector equals the area of the whole circle, which is \( \pi \), times \( \frac{t}{2\pi} \), the ratio of \( t \) to the length of the full circle. The area of the sector is thus \( \frac{t}{2} \), giving us the inequality \( \frac{\tan t}{2} > \frac{t}{2} \), which we can rewrite as \( \frac{\sin t}{t} > \cos t \).

We now have the pair of inequalities

\[
\cos t < \frac{\sin t}{t} < 1.
\]

Since \( \lim_{t \to 0} (\cos t) = 1 \), the relation (2) follows.