Recall that the $n^{th}$ Taylor polynomial for the function $f(x)$ at the point $x = a$ is the polynomial

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$ 

It is the unique polynomial of degree at most $n$ that agrees with $f$ at the point $a$ and whose first $n$ derivatives agree with those of $f$ at the point $a$. With the summation notation it can be rewritten as

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k,$$

with the conventions $f^{(0)} = f$ and $0! = 1$.

The $n^{th}$ Taylor polynomial for $f$ at $a$ not only agrees with $f$ at $a$, also its rate of change at $a$ agrees with that of $f$, and the same is true for the rates of change of the first $n - 1$ derivatives. It is thus reasonable to expect that the Taylor polynomial will approximate $f$ closely for $x$ near $a$. To quantify this expectation one needs an estimate for the error in the approximation.

The difference between $f$ and its $n^{th}$ Taylor polynomial at $a$ is given by

$$R_n(x, a) = f(x) - f(a) - f'(a)(x - a) - \frac{f''(a)}{2}(x - a)^2 - \frac{f'''(a)}{3!}(x - a)^3 - \cdots - \frac{f^{(n)}(a)}{n!}(x - a)^n.$$ 

This is the error in the approximation. Often it is referred to as the remainder in the $n^{th}$ Taylor approximation. How can we obtain a useful estimate of the size of $R_n(x, a)$?

The simplest case is the case $n = 0$, 0th-order Taylor approximation. In this case we are approximating $f$ by the constant function $f(a)$, ridiculous perhaps, but nevertheless indicative of the general case. We have

$$R_0(x, a) = f(x) - f(a) = \int_{ a }^{ x } f'(t)dt.$$ 

If $M$ as an upper bound of $|f'(t)|$ for $t$ between $a$ and $x$, then the preceding integral has absolute value at most $M|x - a|$, i.e., $|R_0(x, a)| \leq M|x - a|$. As we shall see, a similar estimate holds in the general case.

Let’s look at the next simplest case, the case $n = 1$, 1st-order (i.e., linear) approximation. We have

$$R_1(x, a) = f(x) - f(a) - f'(a)(x - a).$$

We now do something clever: instead of keeping the center of approximation $a$ fixed, we let it be variable. We introduce the function

$$R_1(x, t) = f(x) - f(t) - f'(t)(x - t)$$

of the two variables $x$ and $t$. We differentiate $R(x, t)$ with respect to $t$:

$$\frac{\partial R_1(x, t)}{\partial t} = -f'(t) - \frac{\partial}{\partial t}(f'(t)(x - t))$$

$$= -f'(t) - f''(t)(x - t) + f'(t) = -f''(t)(x - t).$$

Now we integrate with respect to $t$ from $a$ to $x$ to get

$$R_1(x, x) - R_1(x, a) = -\int_{ a }^{ x } f''(t)(x - t)dt.$$
But $R(x, x) = 0$, so
\[ R_1(x, a) = \int_a^x f''(t)(x - t)dt. \]

If now $M$ is an upper bound of $|f''(t)|$ between $a$ and $x$, then the absolute value of the preceding integral is at most
\[ M \left| \int_a^x |x - t|dt \right| = \frac{M|x - a|^2}{2}, \]
giving us the error estimate
\[ |R_1(x, a)| \leq \frac{M|x - a|^2}{2}. \]

The preceding method works in the general case to give the following result.

**Theorem.** $R_n(x, a) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n dt$.

The preceding expression is called the integral form of the remainder in Taylor’s formula. Before deriving it, let’s note the following consequence.

**Corollary.** If $|f^{(n+1)}(t)|$ is bounded by $M$ for $t$ between $a$ and $x$, then
\[ R_n(x, a) \leq \frac{M|x - a|^{n+1}}{(n + 1)!}. \]

In fact, if $M$ is an upper bound for $|f^{(n+1)}(t)|$ for $t$ in the interval with endpoints $a$ and $x$, then the integral in the expression for $R_n(x, t)$ is in absolute value no larger than
\[ M \left| \int_a^x |x - t|^n dt \right| = \frac{M|x - a|^{n+1}}{n + 1}. \]

To derive the integral formula for the remainder, we look at the function

\[ R_n(x, t) = f(x) - f(t) - f'(t)(x - t) - \frac{f''(t)}{2} (x - t)^2 - \frac{f^{(3)}(t)}{3!} (x - t)^3 - \cdots - \frac{f^{(n)}(t)}{n!} (x - t)^n \]

of the two variables $x$ and $t$. We take the partial derivative with respect to $t$. There are $n + 2$ summands on the right side of (1). The first of those is $f(x)$, which is independent of $t$, so its derivative with respect to $t$ is 0. The second is $-f(t)$, whose derivative with respect to $t$ is $-f'(t)$. The third is $-f'(t)(x - t)$, whose derivative with respect to $t$ is $-f''(t)(x - t) + f'(t)$. Note that the sum of the derivatives with respect to $t$ of the first three summands is just $-f''(t)(x - t)$ (as we found when treating the case $n = 1$). The fourth summand is $-\frac{f^{(3)}(t)}{2} (x - t)^2$, whose derivative with respect to $t$ is $-\frac{f^{(4)}(t)}{2} (x - t)^2 + f''(t)(x - t)$. The sum of the derivatives with respect to $t$ of the first four summands is thus $-\frac{f^{(3)}(t)}{2} (x - t)^2$. This pattern repeats as we go along. The $(k + 2)^{nd}$ summand is $-\frac{f^{(k)}(t)}{k!} (x - t)^k$, whose derivative with respect to $t$ is
\[ -\frac{f^{(k+1)}(t)}{k!} (x - t)^k + \frac{f^{(k)}(t)}{(k - 1)!} (x - t)^{k-1}. \]
In the derivative of the \((k + 1)^{st}\) summand, the second of these two terms occurs with a minus sign, producing a cancellation. And the first of the two terms appears in the derivative of the \((k + 3)^{rd}\) summand, but with the opposite sign, producing another cancellation — except for \(k = n\), because the \((n + 2)^{nd}\) summand is the last. Adding everything together, we get

\[
\frac{\partial R_n(x, t)}{\partial t} = -\frac{f^{(n+1)}(t)(x - t)^n}{n!}.
\]

Integration now gives

\[
R_n(x, x) - R_n(x, a) = -\frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n dt,
\]

which yields the expression in the theorem since \(R(x, x) = 0\).

The following examples illustrate use of the corollary in estimating errors.

**Example 1.** \(f(x) = \cos x, a = 0\).

The first six Taylor polynomials of \(\cos x\) at the origin are

\[
\begin{align*}
p_0(x) &= p_1(x) = 1 \\
p_2(x) &= p_3(x) = 1 - \frac{x^2}{2} \\
p_4(x) - p_5(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24}.
\end{align*}
\]

All derivatives of \(\cos x\) are bounded in absolute value by 1, so in the corollary we can take \(M = 1\) to get, for example

\[
|R_5(x, 0)| = \left| \cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \right| \leq \left| x \right|^6 = \left| x \right|^6.
\]

When \(x\) is small the error is quite small. For instance,

\[
\left| R_5 \left( \frac{1}{2}, 0 \right) \right| \leq \frac{1}{46080} \leq .000022,
\]

from which we get

\[
\cos \frac{1}{2} \approx 1 - \frac{(1/2)^2}{2} + \frac{(1/2)^4}{24} \approx .87760,
\]

with an error of at most .000022.

**Example 2.** \(f(x) = e^x, a = 0\).

For \(x < 0\) the derivative of \(e^x\) (which is \(e^x\)) is positive and bounded by 1, so, by the corollary,

\[
R_n(x, 0) \leq \frac{|x|^{n+1}}{(n + 1)!}, \quad x < 0.
\]

How large must we take \(n\) to guarantee that \(R_n(-1, 0)\) is at most .0001 in absolute value? We must have \(\frac{1}{(n+1)!} < .0001\), i.e., \((n + 1)! > 10,000\). The smallest such \(n\) is 7 (7! = 5,760, 8! = 40320). The 7th Taylor polynomial of \(e^x\) is

\[
1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}.
\]
Setting \( x = -1 \), then, we see that the number
\[
1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} \approx .367857
\]
is within \( 1/40320 \approx .000025 \) of \( 1/e \).

**Example 3.** \( f(x) = \sqrt{1 + x}, \quad a = 0 \).

Let’s use the fourth Taylor polynomial at the origin to get an estimate for \( \sqrt{3/2} \). We have
\[
\begin{align*}
  f'(x) &= \frac{1}{2}(1 + x)^{-1/2}, \quad f'(0) = \frac{1}{2} \\
  f''(x) &= -\frac{1}{4}(1 + x)^{-3/2}, \quad f''(0) = -\frac{1}{4} \\
  f^{(3)}(x) &= \frac{3}{8}(x + 1)^{-5/2}, \quad f^{(3)}(0) = \frac{3}{8} \\
  f^{(4)}(x) &= -\frac{15}{16}(1 + x)^{-7/2}, \quad f^{(4)}(0) = -\frac{15}{16}
\end{align*}
\]
From this one finds that the fourth Taylor polynomial at the origin is given by
\[
p_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \frac{15}{384}x^4.
\]
Our approximation to \( \sqrt{3/2} \) is thus
\[
p_4 \left( \frac{1}{2} \right) = 1 + \frac{1}{2} \left( \frac{1}{2} \right) - \frac{1}{8} \left( \frac{1}{4} \right) + \frac{3}{48} \left( \frac{1}{8} \right) - \frac{15}{384} \left( \frac{1}{16} \right) \approx 1.22412.
\]
To obtain an error estimate we note that
\[
f^{(5)}(x) = \frac{105}{32} (1 + x)^{-9/2},
\]
which is bounded by \( 105/32 \) for \( x > 0 \). By the corollary,
\[
\left| R_4 \left( \frac{1}{2}, 0 \right) \right| \leq \frac{105}{32} \left( \frac{2^{-5}}{5!} \right) \leq .00086.
\]
Our approximate value for \( \sqrt{3/2} \) is accurate to within .00086.