The method of Lagrange multipliers is often effective in finding solutions of constrained extremum problems. In the two-variable version of such a problem, one is given a function \( f(x, y) \), and one wishes to maximize it or minimize it under the constraint that another function \( g(x, y) \) vanishes (i.e., one wishes to find a maximum or minimum of \( f \) on the level curve \( g(x, y) = 0 \)).

As explained in our textbook (where you will also find examples), Lagrange’s method proceeds as follows. One introduces a third variable \( \lambda \) (traditionally called a Lagrange multiplier), and one defines a function \( F(x, y, \lambda) \) of three variables by

\[
F(x, y, \lambda) = f(x, y) + \lambda g(x, y).
\]

The basic theorem underlying the method states that if \( f(x, y) \) attains a maximum or a minimum at the point \((a, b)\) under the constraint \( g(x, y) = 0 \), then there is a value \( c \) of \( \lambda \) such that \((a, b, c)\) is a critical point of \( F \):

\[
\frac{\partial F}{\partial x}(a, b, c) = 0, \quad \frac{\partial F}{\partial y}(a, b, c) = 0, \quad \frac{\partial F}{\partial \lambda}(a, b, c) = 0.
\]

Thus, in principle, one can find the candidates for the desired constrained extremum of \( f \) by solving the three simultaneous equations (1) for \( a, b, c \). In the nicest situations there will be only one solution, which gives immediately the sought-for extremum \((a, b)\) of \( f \).

The aim here is to explain the geometric underpinning of the method. So assume \( f(x, y) \) does have a maximum or a minimum at \((a, b)\) under the constraint \( g(x, y) = 0 \). We shall assume further that \((a, b)\) is a critical point of neither \( f \) nor \( g \), the most common case. Note first that the partial derivatives of \( F \) are given by

\[
\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial y}{\partial x}, \quad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y}, \quad \frac{\partial F}{\partial \lambda} = g.
\]

The third equality in (1), therefore, just says that \( g(a, b) = 0 \), i.e., that \((a, b)\) satisfies the constraint. The other two equalities in (1) can be written as

\[
\frac{\partial f}{\partial x}(a, b) = -c \frac{\partial g}{\partial x}(a, b), \quad \frac{\partial f}{\partial y}(a, b) = -c \frac{\partial g}{\partial y}(a, b).
\]

What do these mean?

To shorten the notation, let’s define

\[
\alpha = \frac{\partial f}{\partial x}(a, b), \quad \beta = \frac{\partial f}{\partial y}(a, b), \quad \tilde{\alpha} = \frac{\partial g}{\partial x}(a, b), \quad \tilde{\beta} = \frac{\partial g}{\partial y}(a, b).
\]

Rewritten in the new notation, (2) becomes

\[
\alpha = -c \tilde{\alpha}, \quad \beta = -c \tilde{\beta}.
\]

Suppose for definiteness that \((a, b)\) is a maximum of \( f(x, y) \) under the constraint \( g(x, y) = 0 \), and let \( m = f(a, b) \). Consider the level curve \( f(x, y) = m \) (see Figure 3.1). It separates the region where \( f \) is larger than \( m \) from the region where \( f \) is smaller than \( m \). On the level curve \( g(x, y) = 0 \) the function \( f \) takes no value larger than \( m \), so that curve, although it touches the level
curve $f(x, y) = m$ at $(a, b)$, cannot pass through the latter curve; it must stay in the region where $f(x, y) \leq m$. From this it follows that the two curves $f(x, y) = m$ and $g(x, y) = 0$ share a common tangent line at the point $(a, b)$ (see the figure).

The tangent lines at $(a, b)$ to the curves $f(x, y) = m$ and $g(x, y) = 0$ have the respective equations

$$
\alpha(x - a) + \beta(y - b) = 0, \quad \tilde{\alpha}(x - a) + \tilde{\beta}(y - b) = 0
$$

(see Supplementary Notes 1). Now simple algebraic reasoning (left to the reader) shows that the two equations (4) define the same line if and only if the coefficients $\alpha, \beta$ are proportional to the coefficients $\tilde{\alpha}, \tilde{\beta}$, i.e., there is a number $\gamma$ such that $\alpha = \gamma \tilde{\alpha}$ and $\beta = \gamma \tilde{\beta}$. This gives (3) with $c = -\gamma$.

To summarize, the first two equalities in (1) just say that the level curves $f(x, y) = f(a, b)$ and $g(x, y) = 0$ have a common tangent line at the point $(a, b)$. 