HOMEWORK ASSIGNMENT 9

Due in class on Wednesday, November 17.

37. Let the function \( f : (0, 1) \rightarrow \mathbb{R} \) be continuously differentiable. Let the function \( g : (0, 1) \times (0, 1) \rightarrow \mathbb{R} \) be defined by

\[
g(x, y) = \begin{cases} 
\frac{f(x) - f(y)}{x - y}, & x \neq y \\
f'(x), & x = y.
\end{cases}
\]

Prove \( g \) is continuous.

38. Let \( I \) be an open interval and \( f : I \rightarrow \mathbb{R} \) a differentiable function such that \( f' \) is nondecreasing.

(a) Prove \( f' \) is continuous.

(b) Prove \( f \) is convex, i.e., for \( a \) and \( b \) in \( I \) with \( a < b \) and \( 0 < t < 1 \),

\[
f((1 - t)a + tb) \leq (1 - t)f(a) + tf(b).
\]

39. Let \( I \) be an open interval and \( f : I \rightarrow \mathbb{R} \) a twice differentiable function, with \( f'' \) continuous. Prove that, for \( x_0 \) in \( I \),

\[
f''(x_0) = \lim_{\delta \to 0} \frac{f(x_0 + \delta) + f(x_0 - \delta) - 2f(x_0)}{\delta^2}.
\]

40. Let the sequence \((f_n)_{n=1}^{\infty}\) of Riemann integrable functions on the interval \([a, b]\) converge uniformly to the function \( f \). Prove \( f \) is Riemann integrable, and

\[
\int_{a}^{b} f = \lim_{n \to \infty} \int_{a}^{b} f_n.
\]

(Remark: This is proved in Pugh, pp. 207–208, but the proof there uses a deep result of Lebesgue characterizing Riemann integrability which we have not taken up. The result can be proved without resort to Lebesgue’s theorem. You are asked to find such a proof.)