HOMEWORK ASSIGNMENT 7

Due in class on Wednesday, October 27.

27. Let \((a_n)_{n=1}^{\infty}\) be a bounded sequence in \(\mathbb{R}\), with \(\alpha = \liminf_{n \to \infty} a_n\) and \(\beta = \limsup_{n \to \infty} a_n\). Prove \(\limsup_{n \to \infty} a_n^2 = \max\{\alpha^2, \beta^2\}\).

28. Let \((a_n)_{n=1}^{\infty}\) be a convergent sequence and \((b_n)_{n=1}^{\infty}\) a bounded sequence in \(\mathbb{R}\). Prove 
\[
\limsup_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \limsup_{n \to \infty} b_n.
\]

29. Prove the set \(\{x \in \mathbb{R}^3 : \|x\|_2 = 1\}\), the 2-sphere, is connected.

30. Let \(A\) and \(B\) be nonempty subsets of \(\mathbb{R}^M\) and \(\mathbb{R}^N\), respectively. Prove \(A \times B\) is a connected subset of \(\mathbb{R}^{M+N}\) if and only if \(A\) and \(B\) are connected. (Here, we are making the natural identification between \(\mathbb{R}^M \times \mathbb{R}^N\) and \(\mathbb{R}^{M+N}\).)

31. Let \((K_n)_{n=1}^{\infty}\) be a nested sequence of nonempty, compact, connected subsets of a metric space \(M\). Prove the set \(K = \bigcap_{n=1}^{\infty} K_n\) is connected. (Suggestion: Argue by contradiction.)