HOMEWORK ASSIGNMENT 3

Due in class on Wednesday, September 22.

8. Let \((a_n)_{n=1}^{\infty}\) be a convergent sequence of positive real numbers with limit \(b\). Prove that \(\lim_{n \to \infty} \sqrt[n]{a_n} = \sqrt[b]{b}\). (Suggestion: Treat separately the cases \(b = 0\) and \(b > 0\).)

9. (a) Let \((a_n)_{n=1}^{\infty}\) be a convergent sequence of real numbers with limit \(b\). Define the sequence \((b_n)_{n=1}^{\infty}\) by \(b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)\). Prove that \(\lim_{n \to \infty} b_n = b\). (Suggestion: Reduce the general case to the case \(b = 0\).)

(b) Find a divergent sequence \((a_n)_{n=1}^{\infty}\) such that the sequence \((b_n)_{n=1}^{\infty}\) defined as in part (a) is convergent.

10. Let \(t\) be a number in the interval \((0, 1)\). Define the sequence \((a_n)_{n=1}^{\infty}\) by \(a_1 = 1\), \(a_{n+1} = t(a_n + 1)\).

(a) Prove the sequence \((a_n)_{n=1}^{\infty}\) bounded and monotone.

(b) Find \(\lim_{n \to \infty} a_n\).

11. Let \(F_1, F_2, \ldots\) be the Fibonacci numbers, defined by \(F_1 = F_2 = 1\), \(F_{n+2} = F_n + F_{n+1}\). Let \(r_n = \frac{F_{n+1}}{F_n}\).

(a) Prove the sequences \((r_{2n})_{n=1}^{\infty}\) and \((r_{2n-1})_{n=1}^{\infty}\) are monotone.

(b) Prove \((r_n)_{n=1}^{\infty}\) converges, and find its limit.