Counting Points on Igusa Varieties of Hodge Type

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Abstract

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The Langlands-Kottwitz method seeks to understand Shimura varieties in terms of automorphic forms by deriving a trace formula for the cohomology of Shimura varieties which can be compared to the automorphic trace formula. This method was pioneered by Langlands [Lan77, Lan79], and developed further by Kottwitz in [Kot90, Kot92] in the case of Shimura varieties of PEL type with good reduction.

Igusa varieties were introduced in their modern form by Harris-Taylor [HT01] in the course of studying the bad reduction of certain simple Shimura varieties. The relation between Igusa varieties and Shimura varieties was expanded to PEL type by Mantovan [Man04, Man05], and their study of the cohomology of Igusa varieties was expanded to PEL type and streamlined in the Langlands-Kottwitz style by Shin [Shi09, Shi10].

Following the generalization of Mantovan's work to Hodge type by Hamacher and Hamacher-Kim [Ham19, HK19], we carry out the Langlands-Kottwitz method for Igusa varieties of Hodge type, generalizing the work of [Shi09]. That is, we derive a trace formula for the cohomology of Igusa varieties suitable for eventual comparison with the automorphic trace formula.

In order to carry out this method, we also formulate and prove an analogue of the Langlands-Rapoport conjecture for Igusa varieties of Hodge type, building off work of Kisin [Kis17] for Shimura varieties of Hodge and abelian type. Our subsequent arguments on the way to our trace formula are inspired by the techniques of Kisin-Shin-Zhu [KSZ21] using the Langlands-Rapoport conjecture to develop a trace formula for Shimura varieties of Hodge and abelian type.

Dedicated to my parents,
whose curiosity started me off,
and whose support kept me going.

Contents

1 Introduction

1.1 Context

To place this work in historical context, we begin with the Langlands program, which has been a motivating problem in number theory in recent decades. The particular aspect from which this work has arisen is global Langlands reciprocity, which predicts a correspondence between automorphic representations associated to a reductive group G and Galois representations valued in the dual group of G. This correspondence is expected to be realized in the cohomology of Shimura varieties: very roughly, the cohomology of a Shimura variety contains both Galois and automorphic representations, and is expected to decompose into tensor products of a Galois representation and its automorphic partner.

More concretely, a great deal of progress has been made in this direction using the Langlands-Kottwitz method and trace formula techniques. The Langlands-Kottwitz method, pioneered by Langlands [Lan77, Lan79] and developed further by Kottwitz in [Kot90, Kot92], uses a combination of geometric and group-theoretic techniques to obtain a trace formula for the cohomology of Shimura varieties that can be compared to the automorphic trace formula, which comparison eventually allows us to relate Galois and automorphic representations.

This trace formula describing the representations appearing in the cohomology of Shimura varieties is obtained via a fixed-point formula. This changes the problem to working with the points of a Shimura variety mod p, whence the terminology "counting points".

The case treated in [Kot90, Kot92] is that of PEL type and hyperspecial level at p. In this case our Shimura varieties have good reduction and a good moduli structure which makes this approach feasible. More general cases present new challenges.

To see ramified representations we must go beyond hyperspecial level at p, and the resulting Shimura varieties have bad reduction. An approach in the case of modular curves (i.e. GL_2) was described in Deligne's letter to Piatetski-Shapiro. This approach was extended to some Shimura varieties by Harris-Taylor [HT01], where the role of Igusa varieties became clear; and further developed by Mantovan [Man04, Man05] and Shin [Shi09, Shi10, Shi11, Shi12]. In short, Mantovan's formula [Man05, Thm 22] allows us to express the cohomology of Shimura varieties in terms of that of Igusa varieties and Rapoport-Zink spaces, with the bad reduction going to the Rapoport-Zink space and the remaining global information going to the Igusa variety. Then a Langlands-Kottwitz style analysis of Igusa varieties [Shi09, Shi10] allows us to draw conclusions about Shimura varieties [Shi11] and Rapoport-Zink spaces [Shi12].

Beyond PEL type, the construction of integral models of Shimura varieties no longer guarantees a good moduli structure, so more work is needed to get a good description of points in the special fiber. In particular, the Langlands-Rapoport conjecture describes the points on the special fiber of more general Shimura varieties in a way that is suitable for counting points. This conjecture has essentially been proven by Kisin [Kis17] for Shimura varieties of abelian type, and the subsequent point-counting work carried out by Kisin-Shin-Zhu [KSZ21].

Igusa varieties and Mantovan's formula have been generalized to Hodge type by Hamacher and Hamacher-Kim [Ham19, HK19]. The present work is concerned with Igusa varieties of Hodge type. Our goal is to derive a trace formula for the cohomology of Igusa varieties of Hodge type, analogous to those for Shimura varieties given in [Kot90, Kot92, KSZ21] and generalizing the formula for Igusa varieties of PEL type in [Shi09].

1.2 Methods

As we have suggested above, our work is descended from the work of Kisin [Kis17] and Kisin-Shin-Zhu [KSZ21] counting points on Shimura varieties of Hodge (and abelian) type, as well as the work of Shin [Shi09] counting points on Igusa varieties of PEL type. Our methods draw heavily from these sources.

As in the case of Shimura varieties, in Hodge type our Igusa varieties do not have a good moduli structure suitable for counting points. Thus we need a better description of the points on our Igusa variety.

Our first main theorem is an analogue of the Langlands-Rapoport conjecture for Igusa varieties of Hodge type. This is the subject of §3.

Recall (e.g. [Kis17, Conj. 3.3.7]) that the Langlands-Rapoport conjecture for Shimura varieties predicts an equivariant bijection

$$\mathscr{S}_{K_p}(G,X)(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]} I_\phi(\mathbb{Q})\backslash X^p(\phi) \times X_p(\phi),$$

where $\mathscr{S}_{K_p}(G,X)$ is the special fiber of the Shimura variety associated to a datum (G,X) at infinite level away from p; on the right hand side, the objects are defined in terms of Galois gerbs, but intuitively the disjoint union over $[\phi]$ represents the different isogeny classes on the Shimura variety, the sets $X^p(\phi)$ and $X_p(\phi)$ represent away-from-p isogenies and p-power isogenies respectively, and the group $I_\phi(\mathbb{Q})$ represents self-isogenies.

The main difference between Shimura varieties and Igusa varieties is the structure at p, where Igusa varieties fix an isomorphism class of p-divisible group and add the data of a trivialization. Thus to formulate an analogue of the Langlands-Rapoport conjecture for Igusa varieties, we expect this difference to reflect in the set $X_p(\phi)$.

In the case of Shimura varieties we have

$$X_p(\phi) \cong X_v(b) = \{g \in G(L)/G(\mathcal{O}_L) : gb\sigma(g)^{-1} \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)\}$$

(cf. 3.1.1), where v is a cocharacter of G arising from the Shimura datum, and $b \in G(L)$ is an element essentially recording the Frobenius on the isocrystal associated to a chosen point on the Shimura variety (here $L = \breve{\mathbb{Q}}_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$). Our chosen point determines a Dieudonné module inside this isocrystal; intuitively, choosing an element $g \in G(L)/G(\mathcal{O}_L)$ corresponds to transforming this Dieudonné module into another lattice inside the isocrystal, and the condition $gb\sigma(g)^{-1} \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)$ ensures that this lattice is again a Dieudonné module.

For Igusa varieties we replace this $X_v(b)$ by

$$J_b(\mathbb{Q}_p) = \{g \in G(L) : gb\sigma(g)^{-1} = b\}$$

(cf. 2.2.1). Intuitively, replacing the condition $gb\sigma(g)^{-1} \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)$ by the condition $gb\sigma(g)^{-1} = b$ corresponds to fixing an isomorphism class of p-divisible group or Dieudonné module (rather than fixing an isogeny class, i.e. isocrystal); and replacing $G(L)/G(\mathcal{O}_L)$ by $G(L)$ corresponds to adding a trivialization of the p-divisible group (i.e. choosing a basis rather than simply a lattice). We have no need to modify the term $X^p(\phi) \cong G(\mathbb{A}_f^p)$ away from p.

The other change in our analogue of the Langlands-Rapoport conjecture for Igusa varieties is to define a notion of "**b**-admissible morphism" (Definition 3.3.1—here $\mathbf{b} \in B(G)$ is the class of the element b in the paragraphs above) to replace the admissible morphisms appearing in the Langlands-Rapoport conjecture. Since our Igusa variety fixes an isomorphism class of p-divisible group, in particular it fixes an isogeny class, and therefore it lies over a single Newton stratum of the Shimura variety. Restricting to **b**-admissible morphisms corresponds to restricting to isogeny classes in the **b**-stratum.

Indeed, one of the main ideas of the proof (undertaken in §3.2) is to relate isogeny classes on the Igusa variety and the Shimura variety. Namely, we show that taking the preimage of a Shimura isogeny class along the natural map Ig $\to$ Sh gives a bijection between the set of Igusa isogeny classes and the set of Shimura isogeny classes contained in the **b**-stratum (Proposition 3.2.5).

This relation allows us to use the methods of [Kis17] using Kottwitz triples and their refinements to establish a bijection between the set of Igusa isogeny classes and the set of conjugacy classes of **b**-admissible morphisms of Galois gerbs, as well as bijections between each individual isogeny class and its parametrizing set. Note that in [Kis17] the Langlands-Rapoport conjecture is proven only up to possibly twisting the action of $I_\phi(\mathbb{Q})$ on $X^p(\phi) \times X_p(\phi)$ by an element $\tau \in$

$I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$. A crucial part of [KSZ21] is to show that this twist can be taken to satisfy hypotheses that allows us nonetheless to arrive at the expected point-counting formula (namely taking τ to be "tori-rational" and to lie in a distinguished space $\Gamma(\mathcal{H})_0$, cf. 3.5 or [KSZ21, 3.2-3.3]). As we use their methods, the same ambiguity of the twist will appear in our case, but in our case too it will not interfere with point-counting.

Thus we arrive at our first main theorem, an analogue of the Langlands-Rapoport conjecture for Igusa varieties of Hodge type.

Theorem (3.6.2)**.** *There exists a tori-rational element $\tau \in \Gamma(\mathcal{H})_0$ admitting a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]} I_\phi(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p),$$

where the disjoint union ranges over conjugacy classes of $\mathbf{b}$*-admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$, and the action of $I_\phi(\mathbb{Q})$ on $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ is twisted by τ.*

Next we put this theorem to work as our basis for point-counting, to derive the trace formula for cohomology of Igusa varieties of Hodge type. This is the subject of §4.

This falls into two steps. First, we interpret the appropriate class of test functions as correspondences on our Igusa varieties, and use Fujiwara's trace formula to convert the problem of computing traces of the action on cohomology to the problem of computing fixed points of these correspondences. We can then use our first main theorem above to describe the fixed points of these correspondences, resulting in a preliminary form of our point-counting formula parametrized by Galois-gerb-theoretic LR pairs (ϕ, ε) (cf. 4.3.1). This is undertaken in §§4.1-4.2.

For comparison with the automorphic trace formula, the second step is to re-parametrize our point-counting formula in the more group-theoretic terms of Kottwitz parameters (cf. 4.3.8). For this we adapt the techniques of [KSZ21, §§4-5]. The theory required is quite analogous, but the relevant class of LR pairs is different; instead of their "p^n-admissible" pairs, we define notions of "**b**-admissible" and "acceptable" pairs. Then we need to re-work a substantial part of the theory under these new hypotheses. Fortunately it is possible to prove essentially the same results, thought the arguments often require different techniques. This is undertaken in §§4.3-4.5.

In the end we arrive at our second main theorem, the point-counting formula for Igusa varieties of Hodge type.

Theorem (4.5.17)**.** *For any acceptable function $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, we have*

$$\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) =$$
$$\sum_{\gamma_0 \in \Sigma_{\mathbb{R}-\mathrm{ell}}(G)} \sum_{(a,[b_0]) \in \mathcal{KP}(\gamma_0)} \frac{|\text{Ш}_G(\mathbb{Q}, G^\circ_{\gamma_0})|}{|(G_{\gamma_0}/G^\circ_{\gamma_0})(\mathbb{Q})|} \operatorname{vol}\left(I^\circ_{\mathfrak{c}}(\mathbb{Q}) \backslash I^\circ_{\mathfrak{c}}(\mathbb{A}_f)\right) O_{\gamma \times \delta}^{G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)}(f) \operatorname{tr}(\xi(\gamma_0))$$

where $I_{\mathfrak{c}}$ is the inner form of $G^\circ_{\gamma_0}$ associated to the Kottwitz parameter $\mathfrak{c} = (\gamma_0, a, [b_0])$ as in 4.5.3, and γ, δ are the elements belonging to the classical Kottwitz parameter $(\gamma_0, \gamma, \delta)$ associated to $\mathfrak{c}$ as in 4.3.18.

1.3 Applications

In §1.1 we described the motivation for this work from a historical perspective. Along those lines, we expect our formula to be useful in combination with Mantovan's formula to investigate the cohomology of Shimura varieties and Rapoport-Zink spaces, as has been done to great effect in [HT01, Shi11, Shi12].

Another promising application of more recent provenance is to generalize the results of Caraiani-Scholze on cohomology of Shimura varieties [CS17, CS19]. A crucial part of their approach is to push forward along the Hodge-Tate period map $\pi_{\mathrm{HT}} : \mathrm{Sh} \to \mathscr{F}\ell$ and work instead on the flag variety. They realize the fibers of π_{HT} as essentially Igusa varieties, and use the point-counting formula for Igusa varieties PEL type [Shi09, Shi10] to describe the cohomology of those fibers. Thus our second main theorem above is one of the crucial ingredients needed to generalize their arguments to Hodge type.

Even more recently, Kret and Shin [KS21] have given a description of the H^0 cohomology of Igusa varieties in terms of automorphic representations by combining our point-counting formula with automorphic trace formula techniques.

For all these applications, it is necessary to stabilize our point-counting formula. We expect that the methods of [Shi10] can be extended to our case. In addition, we expect that our formula could be generalized to abelian type by making more extensive use of the methods of [KSZ21].

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2 Background

2.1 Isocrystals

In this section we review some theory of p-divisible groups and isocrystals which will be essential for working with Igusa varieties.

2.1.1 Let $L = \breve{\mathbb{Q}}_p$ the completion of the maximal unramified extension of $\mathbb{Q}_p$ and σ the lift of Frobenius on L (coming from $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$). An *isocrystal* over $\overline{\mathbb{F}}_p$ is a finite-dimensional vector space V over L equipped with a σ-semilinear bijection $F : V \to V$, which we call its Frobenius map. A morphism of isocrystals $f : (V_1, F_1) \to (V_2, F_2)$ is a linear map $f : V_1 \to V_2$ intertwining their Frobenius maps, i.e. $f \circ F_1 = F_2 \circ f$.

For G a reductive group, an *isocrystal with G-structure* is an exact faithful tensor functor

$$\mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathrm{Isoc}$$

where Isoc is the category of isocrystals [RR96, Def 3.3]. For $G = \mathrm{GL}_n$, such a functor is determined by its value on the standard representation, and therefore an isocrystal with GL_n-structure is the same as an isocrystal on an n-dimensional vector space.

2.1.2 Giving an isocrystal structure F on L^n is the same as choosing an element $b \in \mathrm{GL}_n(L)$, via $F = b\sigma$. The isocrystal structures defined by $b_0, b_1 \in \mathrm{GL}_n(L)$ are isomorphic precisely when b_0, b_1 are σ-conjugate, i.e. $b_1 = gb_0\sigma(g)^{-1}$ for some $g \in \mathrm{GL}_n(L)$. This gives an identification between isomorphism classes of isocrystals and σ-conjugacy classes in $\mathrm{GL}_n(L)$.

More generally, we can associate an isocrystal with G-structure to an element $b \in G(L)$ by setting "$F = b\sigma$"; namely, the isocrystal with G-structure defined by the functor

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Q}_p}(G) &\to \mathrm{Isoc} \\ (V, \rho) &\mapsto (V \otimes_{\mathbb{Q}_p} L, \rho(b)(\mathrm{id}_V \otimes \sigma)). \end{aligned}$$

This association identifies the set of isomorphism classes of isocrystals with G-structure with the set of σ-conjugacy classes in $G(L)$. We denote this common set by $B(G)$, and we write $[b] \in B(G)$ for the σ-conjugacy class of an element $b \in G(L)$. If G is connected, then in fact every element of $B(G)$ has a representative in $G(\mathbb{Q}_{p^r})$ for some finite unramified extension $\mathbb{Q}_{p^r}$ of $\mathbb{Q}_p$ [Kot85, 4.3]. Given a cocharacter μ of G, there is a distinguished finite subset $B(G, \mu)$ of *μ-admissible* classes, defined in [Kot97, §6].

2.1.3 The Dieudonné-Manin classification gives a concrete description of the category of isocrystals. Given $\lambda = \frac{r}{s}$ a rational number in lowest terms ($s > 0$), we can define an isocrystal $E_\lambda = L\langle F \rangle/(F^s - p^r)$ with Frobenius given by left multiplication by F; here $L\langle F \rangle$ is the twisted polynomial ring where $Fx = \sigma(x)F$ for $x \in L$. The Dieudonné-Manin classification states that the category of isocrystals is semi-simple, and the simple objects are E_λ for $\lambda \in \mathbb{Q}$. In other words, an isocrystal is determined by a finite set of rational numbers λ with multiplicities. These rational numbers are called its *slopes*, and the decomposition of an isocrystal into a direct sum $V = \bigoplus_\lambda V_\lambda$ of subspaces of slope λ (so $V_\lambda \cong E_\lambda^{\oplus r}$ for some r) is called the *slope decomposition*.

We can also associate slopes to an isocrystal with G-structure, in a slightly different form. Let $\mathbb{D}$ be the pro-torus with character group $X^*(\mathbb{D}) = \mathbb{Q}$. Then an isocrystal with slope decomposition $V = \bigoplus_\lambda V_\lambda$ produces a fractional cocharacter $\mathbb{D} \to \mathrm{GL}(V)$ over L defined by $\mathbb{D}$ acting on V_λ by the character $\lambda \in \mathbb{Q} = X^*(\mathbb{D})$.

Now let $b \in G(L)$, defining an isocrystal with G-structure. Given a representation $(V, \rho) \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$, this isocrystal with G-structure produces an isocrystal on $V_L = V \otimes_{\mathbb{Q}_p} L$, and therefore a fractional cocharacter $\nu_\rho : \mathbb{D} \to \mathrm{GL}(V_L)$. The *slope homomorphism* of b is the unique fractional cocharacter $\nu_b : \mathbb{D} \to G$ over L satisfying $\nu_\rho = \rho \circ \nu_b$ for all p-adic representations ρ of G.

Alternatively, ν_b can be defined (cf. [Kot85, 4.3]) as the unique element of $\mathrm{Hom}_L(\mathbb{D}, G)$ for which there exists an $n > 0$ and $c \in G(L)$ such that

- $n\nu_b \in \mathrm{Hom}_L(\mathbb{G}_m, G)$,
- $\mathrm{Int}(c) \circ n\nu_b$ is defined over a finite unramified extension $\mathbb{Q}_{p^n}$ of $\mathbb{Q}_p$, and
- $c(b\sigma)^n c^{-1} = c \cdot n\nu_b(\pi) \cdot c^{-1} \cdot \sigma^n$ (considered in $G(L) \rtimes \langle \sigma \rangle$).

From the definition we can see the slope homomorphism transforms nicely under the action of σ and conjugation by $G(L)$:

- $\nu_{\sigma(b)} = \sigma(\nu_b)$;
- $\nu_{gb\sigma(g)^{-1}} = \mathrm{Int}(g) \circ \nu_b$.

The following lemma states that, to check if an element $g \in G(L)$ commutes with ν_b, it suffices to check on a single faithful representation.

Lemma 2.1.4. *Let $b \in G(L)$, defining an isocrystal with G-structure. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful p-adic representation, and $(V_L, \rho(b)\sigma)$ the associated isocrystal. If $g \in G(L)$ (acting via $\rho(g)$) preserves the slope decomposition of this isocrystal, then g commutes with the slope homomorphism ν_b.*

Proof. Suppose $g \in G(L)$ preserves the slope decomposition $V_L = \bigoplus_\lambda V_\lambda$. Consider the two fractional cocharacters ν_b and $\mathrm{Int}(g) \circ \nu_b$ of G; we want to show they are equal.

Composing with ρ, we get two fractional cocharacters of $\mathrm{GL}(V_L)$,

$$\nu_\rho = \rho \circ \nu_b \qquad \text{and} \qquad \mathrm{Int}(\rho(g)) \circ \nu_\rho = \rho \circ \mathrm{Int}(g) \circ \nu_b.$$

The fractional cocharacter $\nu_\rho : \mathbb{D} \to \mathrm{GL}(V_L)$ is defined by $\mathbb{D}$ acting on V_λ by the character $\lambda \in \mathbb{Q} = X^*(\mathbb{D})$. By assumption $\rho(g)$ preserves the slope decomposition, so we see that $\mathrm{Int}(\rho(g)) \circ \nu_\rho$ also acts on V_λ by the character λ. Since these two actions are the same, we find $\nu_\rho = \mathrm{Int}(\rho(g)) \circ \nu_\rho$; and by the monomorphism property of ρ this implies $\nu_b = \mathrm{Int}(g) \circ \nu_b$, as desired. □

2.2 Acceptable Elements of $J_b(\mathbb{Q}_p)$

In this section we define the group J_b and the notion of an acceptable element, which will play a central role in our analysis.

2.2.1 For $b \in G(L)$, define an algebraic group J_b (or J_b^G when it is helpful to specify the group) over $\mathbb{Q}_p$ by defining its points for a $\mathbb{Q}_p$-algebra R by

$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} L) : gb\sigma(g)^{-1} = b\},$$

and define

$$M_b = \text{centralizer in } G \text{ of } \nu_b.$$

If necessary we can change b inside its σ-conjugacy class to ensure that M_b is defined over $\mathbb{Q}_p$ (this is always possible if G is quasi-split, cf. [Kot85, Prop 6.2]). Then J_b is the automorphism group of the isocrystal with G-structure defined by b (indeed, the condition $gb\sigma(g)^{-1} = b$ precisely means that g commutes with $b\sigma$), and furthermore J_b is an inner form of M_b. Changing b by σ-conjugation in $G(L)$ does not essentially change the situation: if $b_0 = gb_1\sigma(g)^{-1}$, then we have a canonical isomorphism

$$\begin{aligned} J_{b_1} &\xrightarrow{\sim} J_{b_0} \\ x &\longmapsto gxg^{-1} \end{aligned}$$

and $M_{b_0} = \mathrm{Int}(g) M_{b_1}$.

2.2.2 To define acceptable elements, we work with a concrete isocrystal (rather than the isocrystal with G-structure as a functor). The definition does not depend on our choices, but the choices make things simpler. And in our case, we will be working in Hodge type with a fixed Hodge embedding $G \hookrightarrow \mathrm{GSp}(V,\psi)$ which furnishes a fixed faithful representation of G, so we do not rely on the independence of choice.

With those caveats, choose a faithful representation V of G. Then our isocrystal with G-structure associated to b produces an isocrystal $(V_L, b\sigma)$ (we abuse notation by suppressing the map ρ, e.g. writing $b\sigma$ rather than $\rho(b)\sigma$ for the Frobenius). The group $J_b(\mathbb{Q}_p)$ acts on this isocrystal by linear automorphisms, via its natural inclusion in $G(L)$—the defining condition $gb\sigma(g)^{-1} = b$ exactly means that g commutes with $b\sigma$. Write $V_L = \bigoplus_i V_{\lambda_i}$ for the slope decomposition of our isocrystal, with slopes in decreasing order $\lambda_1 > \lambda_2 > \cdots > \lambda_r$.

Definition 2.2.3. Define an element $\delta \in J_b(\mathbb{Q}_p)$ to be *acceptable* (or say δ is *acceptable with respect to* b) if, regarding $\delta = (\delta_i) \in \prod \mathrm{GL}(V_{\lambda_i})$, any eigenvalues e_i of δ_i and e_j of δ_j with $i < j$ (i.e. $\lambda_i > \lambda_j$) satisfy $v_p(e_i) < v_p(e_j)$.

That is, the eigenvalues of δ on slope components of our isocrystal are separated by p-adic valuation.

An important example of an acceptable element is defined as follows. Choose an integer s so that $s\lambda_i$ is an integer for all slopes λ_i of our isocrystal. Define an element of $J_b(\mathbb{Q}_p)$, formally written as fr^s, by acting on V_{λ_i} by $p^{s\lambda_i}$. For the inverse of this element we write fr^{-s}, defined by acting on V_{λ_i} by $p^{-s\lambda_i}$. Intuitively, we think of these elements as Frobenius to the power s (resp. $-s$)

The next lemma verifies that acceptable elements are plentiful enough for our later purposes.

Lemma 2.2.4. *Let $S \subset J_b(\mathbb{Q}_p)$ a compact subset. Then for a sufficiently large power n, the set $(fr^{-s})^n S$ consists of acceptable elements.*

Proof. Using the notation of Definition 2.2.3, let $\delta \in J_b(\mathbb{Q}_p)$, and let α_i be an eigenvalue of δ_i. Multiplying δ by fr^{-s} has the effect of multiplying α_i by $p^{-s\lambda_i}$, which decreases its p-adic valuation by $s\lambda_i$. For $\lambda_i > \lambda_j$, this decreases the p-adic valuation of eigenvalues of δ_i relative to those of δ_j, so by taking a high enough power n we can arrange that $(fr^{-s})^n\delta$ is acceptable. Since eigenvalues vary continuously and p-adic valuation is continuous, the compactness of S implies that the p-adic valuations of all eigenvalues of elements of S lie in a compact subset of $\mathbb{R}_{>0}$, so we can find a uniform power of fr^{-s} over the whole subset S. □

The next two results use the acceptable condition to get technical conditions that will be of use later.

Lemma 2.2.5. *Let $\varepsilon \in G(\mathbb{Q}_p^{\mathrm{ur}})$ semi-simple. Suppose ε commutes with $b\sigma$, and is therefore an element of $J_b(\mathbb{Q}_p)$, and furthermore is acceptable. Then $G_\varepsilon \subset M_b$.*

Proof. We continue to work with our fixed isocrystal of 2.2.2, with slope decomposition

$$V_L = \bigoplus_i V_{\lambda_i}.$$

Since ε is semi-simple, its action on V_L is diagonalizable, and since it commutes with $b\sigma$ it preserves the slope components V_{λ_i}. Thus each slope component has a basis of eigenvectors for the action of ε. The acceptable condition implies that ε has different eigenvalues on different slope components, so in fact each slope component is a direct sum of full eigenspaces of ε.

Now, suppose $x \in G_\varepsilon$, so x commutes with ε. Then x preserves the eigenspaces of ε. Since the slope components are direct sums of eigenspaces of ε, we see that x preserves the slope decomposition. By Lemma 2.1.4 this implies $x \in M_b$. □

Lemma 2.2.6. *Let b_0, b_1 be σ-conjugate elements of $G(L)$. Suppose that there is a semi-simple element $\varepsilon \in G(L)$ such that ε lies in both $J_{b_0}(\mathbb{Q}_p)$ and $J_{b_1}(\mathbb{Q}_p)$ and furthermore is acceptable with respect to both b_0 and b_1. Then $\nu_{b_0} = \nu_{b_1}$.*

In other words, we know that two σ-conjugate elements have conjugate slope homomorphisms—the Lemma claims that if they have a (semi-simple) acceptable element in common, their slope homomorphisms are in fact equal.

Proof. The idea is to consider the slope decompositions of the isocrystals defined by b_0 and b_1—the σ-conjugacy of b_0, b_1 allows us to identify the slope decompositions in one way, and the acceptable property allows us to identify them in another way, and together they give us the result.

We continue to work with our fixed representation V of 2.2.2, and write $(V_L, b_0\sigma)$ and $(V_L, b_1\sigma)$ for the isocrystals produced from V by b_0 and b_1. These two isocrystal structures induce two slope decompositions

$$V_L = \bigoplus_i V_{\lambda_i,0} \qquad \text{and} \qquad V_L = \bigoplus_i V_{\lambda_i,1}.$$

Since b_0, b_1 are σ-conjugate, we see these two isocrystal structures are abstractly isomorphic. Taking $g \in G(L)$ with $b_1 = g b_0 \sigma(g)^{-1}$, the slope decompositions are related by $V_{\lambda_i,1} = g \cdot V_{\lambda_i,0}$. In particular, for each slope λ_i, the slope components $V_{\lambda_i,0}$ and $V_{\lambda_i,1}$ have the same dimension.

Now consider the element ε, which we assume to commute with $b_0\sigma, b_1\sigma$ so that it acts on both isocrystal structures, and furthermore is acceptable for both. As in the proof of 2.2.5, the semi-simple and acceptable conditions imply that each slope component $V_{\lambda_\bullet,\bullet}$ is a direct sum of full eigenspaces of ε. Combining the fact that $\dim V_{\lambda_i,0} = \dim V_{\lambda_i,1}$ with the fact that (by the acceptable condition) the eigenspaces appearing in the various slope components must be ordered by the p-adic valuation of the corresponding eigenvalue, this shows that $V_{\lambda_i,0}$ and $V_{\lambda_i,1}$ must consist of the same eigenspaces; that is, $V_{\lambda_i,0} = V_{\lambda_i,1}$ for all slopes λ_i.

Now we've shown $g \cdot V_{\lambda_i,0} = V_{\lambda_i,1} = V_{\lambda_i,0}$; that is, g preserves the slope decomposition with respect to b_0, and by Lemma 2.1.4 this means g centralizes ν_{b_0}. On the other hand, we chose g to satisfy $b_1 = gb_0\sigma(g)^{-1}$, so as in 2.1.3 we have

$$\nu_{b_1} = \nu_{gb_0\sigma(g)^{-1}} = \mathrm{Int}(g) \circ \nu_{b_0}.$$

Thus $\nu_{b_0} = \nu_{b_1}$, as desired. □

2.2.7 We now define a submonoid $S_b \subset J_b(\mathbb{Q}_p)$ which will play a role in defining the group actions on our Igusa varieties. Let $\delta \in J_b(\mathbb{Q}_p)$, and suppose that δ^{-1} is an isogeny. Regard $\delta = (\delta_i) \in \prod \mathrm{GL}(V_{\lambda_i})$ as in 2.2.2–2.2.3, and for each i let $e_i(\delta)$ and $f_i(\delta)$ be the minimal and maximal (respectively) integers such that

$$\ker p^{f_i(\delta)} \subset \ker \delta_i^{-1} \subset \ker p^{e_i(\delta)}.$$

Then S_b is the submonoid of $J_b(\mathbb{Q}_p)$ defined by

$$S_b = \{\delta \in J_b(\mathbb{Q}_p) : \delta^{-1} \text{ an isogeny, and } f_{i-1}(\delta) \geq e_i(\delta) \text{ for all } i\}.$$

For example, S_b contains p^{-1}, as multiplication by p is an isogeny and $f_{i-1}(p^{-1}) = e_i(p^{-1}) = 1$ for all i. Also S_b contains fr^{-s}, as fr^s is an isogeny and $f_{i-1}(fr^{-s}) = s\lambda_{i-1} > e_i(fr^{-s}) = s\lambda_i$ for all i. In fact, $J_b(\mathbb{Q}_p)$ is generated as a monoid by S_b together with p and fr^s—in other words, any element of $J_b(\mathbb{Q}_p)$ can be translated into S_b by multiplying by high enough powers of p^{-1} and fr^{-s}.

2.3 p-divisible Groups

In this section we briefly recall some definitions related to p-divisible groups that will be essential for defining and working with Igusa varieties.

2.3.1 A *p-divisible group* over a scheme S is an fppf sheaf of abelian groups $\mathscr{G}$ on S such that

- $\mathscr{G}[p^n] = \ker(\mathscr{G} \xrightarrow{p^n} \mathscr{G})$ is a finite locally free group scheme for all n,
- $\mathscr{G} = \varinjlim_n \mathscr{G}[p^n]$, and
- $\mathscr{G} \xrightarrow{p} \mathscr{G}$ is an epimorphism.

An *isogeny* of p-divisible groups $f : \mathscr{G}_1 \to \mathscr{G}_2$ is an epimorphism whose kernel is a finite locally free group scheme. A *quasi-isogeny* is a global section f of $\mathrm{Hom}_S(\mathscr{G}_1, \mathscr{G}_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ which Zariski-locally admits an integer n so that $p^n f$ is an isogeny.

The examples we will be most concerned with are p-divisible groups arising from abelian varieties: if A is an abelian variety, then the limit $A[p^\infty] = \varinjlim A[p^n]$ over the natural inclusion maps $A[p^n] \hookrightarrow A[p^{n+1}]$ forms a p-divisible group.

We write $\mathscr{G} \mapsto \mathbb{D}(\mathscr{G})$ for the contravariant Dieudonné module functor, which gives a contravariant equivalence of categories between the category of p-divisible groups over $\overline{\mathbb{F}}_p$ and the category of Dieudonné modules (e.g. [Dem72]). By composing with the functor from Dieudonné modules to isocrystals, we get a contravariant functor $\mathscr{G} \to \mathbb{V}(\mathscr{G})$.

While the Dieudonné module records a p-divisible group up to isomorphism, the isocrystal records a p-divisible group up to isogeny. To be precise, consider the *isogeny category* of p-divisible groups over S, whose objects are p-divisible groups and whose morphisms are global sections of $\mathrm{Hom}_S(\mathscr{G}_1, \mathscr{G}_2) \otimes_{\mathbb{Z}} \mathbb{Q}$; in this category quasi-isogenies are isomorphisms. Then the functor $\mathscr{G} \to \mathbb{V}(\mathscr{G})$ is an equivalence of categories between the isogeny category of p-divisible groups over $\overline{\mathbb{F}}_p$ and the category of isocrystals.

A p-divisible group can be equipped with G structure; in our case this will usually consist of a set of tensors on $\mathbb{D}(\mathscr{G})$. Then the associations above produce a Dieudonné module with G-structure and an isocrystal with G-structure. In particular a p-divisible group with G-structure has an associated isocrystal with G-structure, whose isomorphism class is recorded by a class $[b] \in B(G)$, and we say the p-divisible group is of *type* $[b]$.

2.3.2 In order to define Igusa varieties later we introduce the "completely slope divisible" property on p-divisible groups.

Say that a p-divisible group is *isoclinic* if it has only a single slope (possibly with multiplicity). A *slope filtration* for a p-divisible group $\mathscr{G}$ with slopes $\lambda_1 > \cdots > \lambda_r$ is a filtration

$$0 = \mathscr{G}_0 \subset \cdots \subset \mathscr{G}_r = \mathscr{G}$$

such that each successive quotient $\mathscr{G}_i / \mathscr{G}_{i-1}$ is isoclinic of slope λ_i. If it exists, it is unique. A slope filtration always exists for a p-divisible group over a field of positive characteristic, and the filtration splits canonically if the field is perfect [Gro74].

We say $\mathscr{G}$ is *completely slope divisible* if it has a slope filtration such that for each successive quotient X of slope $\lambda = \frac{a}{b}$, the quasi-isogeny $p^{-a}\,\mathrm{Frob}^b : X \to X^{(p^b)}$ is an isogeny. Over $\overline{\mathbb{F}}_p$, this is equivalent to being a direct sum of isoclinic p-

divisible groups defined over finite fields [OZ02]. For a class $[b] \in B(G)$, there is a completely slope divisible p-divisible group with G-structure of type $[b]$ exactly when $[b]$ has a representative in $G(\mathbb{Q}_{p^r})$ for some r; as in 2.1.2, this is the case for all classes in $B(G)$ if $G_{\mathbb{Q}_p}$ is connected.

2.4 Igusa Varieties of Siegel Type

In this section we review the case of Siegel type, as it is required for understanding Hodge type. Many objects will be decorated with a tick $\bullet'$ to distinguish them from the analogous objects of Hodge type introduced in §2.5.

2.4.1 Let V be a $\mathbb{Z}$-module of finite rank $2r$ and ψ a symplectic pairing on V. For any ring R, we write $V_R = V \otimes_{\mathbb{Z}} R$. This data gives rise to a Shimura datum of Siegel type $(\mathrm{GSp}, S^{\pm})$ consisting of the group of symplectic similitudes

$$\mathrm{GSp}(R) = \{g \in \mathrm{GL}(V_R) : \psi(gx, gy) = c(g)\psi(x, y) \text{ for some } c(g) \in R^{\times}\}$$

and the Siegel double space $S^{\pm}$, defined to be the set of complex structures J on $V_{\mathbb{R}}$ such that the form $(x, y) \mapsto \psi(x, Jy)$ is symmetric and positive or negative definite. We can regard such a complex structure on $V_{\mathbb{R}}$ also as a Hodge structure on V, or also (in the usual manner of a Shimura datum) as a homomorphism $\mathbb{S} \to \mathrm{GSp}_{\mathbb{R}}$, where $\mathbb{S}$ denotes the Deligne torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.

We can consider GSp as a reductive group over $\mathbb{Z}_{(p)}$, acting on $V_{\mathbb{Z}_{(p)}}$, and let $K'_p = \mathrm{GSp}(\mathbb{Z}_p) \subset \mathrm{GSp}(\mathbb{Q}_p)$ be the corresponding hyperspecial subgroup. We write $K' = K'_p K'^p$, where $K'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup.

The Shimura variety $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ associated to this Shimura datum is defined over $\mathbb{Q}$, and has a canonical integral model which can be defined by a moduli problem. Consider the category of abelian schemes up to prime-to-p isogeny, whose objects are abelian schemes, and whose morphisms are $\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ (where $\mathrm{Hom}(A, B)$ denotes the usual homomorphisms of abelian schemes). Isomorphisms in this category are called prime-to-p quasi-isogenies. Our moduli problem has the form

$$\begin{aligned} \textbf{Schemes}_{/\mathbb{Z}_{(p)}} &\longrightarrow \textbf{Sets} \\ X &\longmapsto \{(A, \lambda, \eta^p_{K'})\}/\sim \end{aligned}$$

where

- A is an abelian scheme over X up to prime-to-p isogeny;

- λ is a weak polarization of A, i.e. a prime-to-p quasi-isogeny $\lambda : A \to A^\vee$ modulo scaling by $\mathbb{Z}_{(p)}^\times$, some multiple of which is a polarization;

- $\eta^p_{K'} \in \Gamma(X, \underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \hat{V}^p(A))/K'^p)$ is a K'^p-level structure, where we regard $\hat{T}^p(A) = \varprojlim_{p \nmid n} A[n]$ and $\hat{V}^p(A) = \hat{T}^p(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ as étale sheaves on X, and define $\underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \hat{V}^p(A))$ to be the étale sheaf of isomorphisms compatible with the pairings induced by ψ and λ up to $\mathbb{A}_f^{p\times}$-scalar; and

- two triples are equivalent $(A_1, \lambda_1, \eta^p_{K',1}) \sim (A_2, \lambda_2, \eta^p_{K',2})$ if there is a prime-to-p quasi-isogeny $A_1 \to A_2$ sending $\lambda_1 \to \lambda_2$ and $\eta^p_{K',1}$ to $\eta^p_{K',2}$.

If K'^p is sufficiently small, this moduli problem is represented by a smooth $\mathbb{Z}_{(p)}$-scheme $\mathscr{S}_{K'}(\mathrm{GSp}, S^\pm)$, which is the canonical integral model of $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^\pm)$. By virtue of the moduli structure, it carries a universal polarized abelian scheme $\mathcal{A}' \to \mathscr{S}_{K'}(\mathrm{GSp}, S^\pm)$. Write

$$\overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^\pm) = \mathscr{S}_{K'}(\mathrm{GSp}, S^\pm) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$$

for the special fiber.

2.4.2 The universal polarized abelian scheme gives rise to a universal polarized p-divisible group $(\mathcal{A}'[p^\infty], \lambda)$ and hence isocrystal with GSp-structure over $\overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^\pm)$. Restricting to a geometric point $\overline{x} \to \overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^\pm)$ gives an isocrystal with GSp-structure over $\overline{x}$, which is classified by an element $\mathbf{b}_x \in B(\mathrm{GSp})$ depending only on the topological point x underlying $\overline{x}$. For each $\mathbf{b} \in B(\mathrm{GSp})$, we define the *Newton stratum*

$$\overline{\mathscr{S}}^{(\mathbf{b})}_{K'}(\mathrm{GSp}, S^\pm) = \{x \in \overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^\pm) : \mathbf{b}_x = \mathbf{b}\}$$

to be the locus in $\overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^\pm)$ where the universal p-divisible group is of type $\mathbf{b}$. It is a locally closed subset, which we promote to a subscheme by taking the reduced subscheme structure.

Fixing a class $\mathbf{b} \in B(\mathrm{GSp})$ whose Newton stratum is non-empty, let (Σ, λ_Σ) be a p-divisible group with GSp-structure of type $\mathbf{b}$, i.e.

- Σ a p-divisible group over $\overline{\mathbb{F}}_p$ and

- λ_Σ a polarization of Σ, such that

- there is an isomorphism $\mathbb{D}(\Sigma) \xrightarrow{\sim} V_{\mathcal{O}_L}$ preserving the pairings induced by λ_Σ and ψ, and taking the Frobenius on $\mathbb{D}(\Sigma)$ to an endomorphism $b\sigma$ on $V_{\mathcal{O}_L}$ with $b \in \mathrm{GSp}(\mathcal{O}_L)$ belonging to the class $\mathbf{b} \in B(\mathrm{GSp})$.

Note choosing a different isomorphism changes b by σ-conjugacy in $\mathrm{GSp}(\mathcal{O}_L)$, so its class in $B(\mathrm{GSp})$ is well-defined.

A Newton stratum is the locus obtained by fixing an isogeny class of p-divisible group; we further decompose the special fiber of our Shimura variety by fixing an isomorphism class of p-divisible group. Define the *central leaf* corresponding to (Σ, λ_Σ) by

$$C'_{\Sigma,K'} = \{x \in \overline{\mathscr{S}}^{(\mathbf{b})}_{K'}(\mathrm{GSp}, S^\pm) : (\mathcal{A}[p^\infty]_x, \lambda_x) \otimes_{k(x)} \overline{k(x)} \cong (\Sigma, \lambda_\Sigma) \otimes_{\overline{\mathbb{F}}_p} \overline{k(x)}\}.$$

This is a closed subset of the Newton stratum, and when equipped with the reduced subscheme structure, is smooth [Man05, Prop 1].

2.4.3 Now we assume that Σ is completely slope divisible (which we can do since GSp is connected, cf. 2.3.2). Then the universal p-divisible group $\mathcal{A}'[p^\infty]$ over C_Σ, being isomorphic to Σ (over each geometric generic point of C_Σ), is also completely slope divisible. Let $\mathcal{A}'[p^\infty]^{(i)}$ be the successive quotients of the slope filtration, and define $\mathcal{A}'[p^\infty]^{\mathrm{sp}} = \bigoplus_i \mathcal{A}'[p^\infty]^{(i)}$ to be the associated split p-divisible group, which inherits a polarization from $\mathcal{A}'[p^\infty]$. Denote its p^m-torsion by $\mathcal{A}'[p^m]^{\mathrm{sp}}$.

The *level-m Igusa variety* of Siegel type $\mathrm{Ig}'_{\Sigma,K',m}$ is a smooth $\overline{\mathbb{F}}_p$-scheme, finite étale and Galois over $C'_{\Sigma,K'}$, defined by the moduli problem

$$\mathrm{Ig}'_{\Sigma,K',m}(X) = \{(A, \lambda, \eta^p_{K'}, j_m) : (A, \lambda, \eta^p_{K'}) \in C'_{\Sigma,K'}(X),\ j_m : \Sigma[p^m] \times_{\overline{\mathbb{F}}_p} X \xrightarrow{\sim} \mathcal{A}'[p^m]^{\mathrm{sp}} \times_{C_\Sigma} X\},$$

where j_m is an isomorphism preserving polarizations up to $(\mathbb{Z}/p^m)^\times$-scalar, and extending étale locally to any higher level $m' \geq m$. Essentially we are adding level structure over $C'_{\Sigma,K'}$ in the form of a trivialization j_m of $\mathcal{A}'[p^m]$.

Let $\mathrm{Ig}'_{\Sigma,K'} = \varprojlim_m \mathrm{Ig}'_{\Sigma,K',m}$ be the Igusa variety at infinite m-level, and $\mathscr{J}'_{\Sigma,K'} = \mathrm{Ig}'^{(p^{-\infty})}_{\Sigma,K'}$ its perfection. Then $\mathscr{J}'_{\Sigma,K'}$ is the moduli space over $C'_{\Sigma,K'}$ parametrizing isomorphisms

$$j : \Sigma \times_{\overline{\mathbb{F}}_p} C'_{\Sigma,K'} \to \mathcal{A}'[p^\infty]$$

preserving polarizations up to scaling. Note that the slope filtration splits canonically over a perfect base [Man, 4.1], so we no longer need to impose the splitting on $\mathcal{A}'[p^\infty]$.

The group $\mathrm{GSp}(\mathbb{A}^p_f)$ acts on the system $\mathrm{Ig}'_{\Sigma,K'}$, varying K', by acting on the level structure $\eta^p_{K'}$. This action is inherited from the Siegel modular variety, since it happens away from p (i.e. it does not interact with the Igusa level structure). The system $\mathrm{Ig}'_{\Sigma,K'}$ also receives an action of the submonoid $S_b \subset J^{\mathrm{GSp}}_b(\mathbb{Q}_p)$ of 2.2.7, which extends to an action of the full group $J^{\mathrm{GSp}}_b(\mathbb{Q}_p)$ on the perfections $\mathscr{J}'_{\Sigma,K'}$ and on étale cohomology.

2.5 Igusa Varieties of Hodge Type

We now turn to the case of Hodge type. To define Igusa varieties of Hodge type, we need many of the same constructions as in Siegel type §2.4, but they require slightly different techniques.

2.5.1 Let (G, X) be a Shimura datum of Hodge type: G is a connected reductive $\mathbb{Q}$-group, which we further assume to be unramified at p; X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$; and there exists a closed embedding $G \hookrightarrow \mathrm{GSp}$ which sends X to $S^{\pm}$. Denote by E the reflex field of (G, X).

We permanently fix a Hodge embedding $G \hookrightarrow \mathrm{GSp}$. In particular, we will regard G as having a fixed action on V via this fixed embedding, and we will abuse notation by giving the same name to an element of G and its image in GSp or action on V.

As G is unramified, it has a reductive model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ and corresponding hyperspecial subgroup $K_p = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$. As in [Kis17, 1.3.3], there is a $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V_{\mathbb{Q}}$ such that the embedding $G \hookrightarrow \mathrm{GSp}$ is induced by an embedding $G_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$. Enlarging our symplectic space V if necessary, ψ induces a perfect pairing on $V_{\mathbb{Z}(p)}$, so we can define a hyperspecial subgroup $K'_p = \mathrm{GSp}(V_{\mathbb{Z}_{(p)}})(\mathbb{Z}_p) \subset \mathrm{GSp}(\mathbb{Q}_p)$ which is compatible in the sense that the embedding $G \hookrightarrow \mathrm{GSp}$ takes K_p into K'_p. For any compact open $K^p \subset G(\mathbb{A}_f^p)$ there is a $K'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ so that $K = K_p K^p \subset K' = K'_p K'^p$ and the natural map

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$$

is a closed embedding.

In the case of Hodge type, our integral models will not be defined by a moduli problem. Instead, consider the composition

$$\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm}) \to \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm}),$$

and let $\mathscr{S}_K(G, X)$ be the closure of $\mathrm{Sh}_K(G, X)$ in $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(p)}$ (where $\mathcal{O}_{E,(p)} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$). Then $\mathscr{S}_K(G, X)$ is the canonical integral model of $\mathrm{Sh}_K(G, X)$. (By [Kis10], the normalization of $\mathscr{S}_K(G, X)$ is the canonical integral model, and recently it has been shown [Xu20] that the normalization is unnecessary).

Pulling back the universal abelian scheme $\mathscr{A}' \to \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$ along the map $\mathscr{S}_K(G, X) \to \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$ we obtain a universal abelian scheme $\mathcal{A} \to \mathscr{S}_K(G, X)$ and p-divisible group $\mathcal{A}[p^{\infty}]$.

2.5.2 To make up for the lack of honest moduli structure, we equip our objects with G-structure essentially by hand, in the form of tensors. For a vector space or module W, we write $W^{\otimes}$ be the direct sum of all finite combinations of tensor powers, duals, and symmetric and exterior powers of W. Since we include duals, we can identify $W^{\otimes}$ with $(W^{\vee})^{\otimes}$.

As in [Kis10, 2.3.2], our group G has a reductive model $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ defined as a subgroup of $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ by a finite collection of tensors $\{s_\alpha\} \subset V_{\mathbb{Z}_{(p)}}^{\otimes}$. Fix such a collection of tensors $\{s_\alpha\}$.

Our G-structures essentially amount to transporting the tensors $\{s_\alpha\}$ to all relevant spaces. Following [Kis17, 1.3.6-10], to each point $x \in \mathscr{S}_K(G,X)(S)$ we associate a finite set of tensors

$$\{s_{\alpha,\ell,x}\} \subset H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell)^{\otimes} \cong V_\ell(\mathcal{A}_x)^{\otimes}, \qquad \ell \neq p$$

(here we use the insensitivity of $\bullet^{\otimes}$ to dualizing); and to each point $x \in \mathscr{S}_K(G,X)(\overline{\mathbb{F}}_p)$ a finite set of tensors

$$\{s_{\alpha,0,x}\} \subset H^1_{\text{crys}}(\mathcal{A}_x/\mathcal{O}_L)^{\otimes} \cong \mathbb{D}(\mathcal{A}_x[p^\infty])^{\otimes}$$

(defined even over $\mathbb{Z}_{p^r}$ for r sufficiently large). These tensors are compatible with the original tensors $\{s_\alpha\}$ in the following way. As in [Kis10, 3.2.4], for $x \in \mathscr{S}_K(G,X)(S)$ with associated Siegel data $(\mathcal{A}_x, \lambda, \eta^p_{K'})$, the section

$$\eta^p_{K'} \in \Gamma(S, \underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \hat{V}^p(\mathcal{A}_x))/K'^p)$$

can be promoted to a section

$$\eta^p_K \in \Gamma(S, \underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p}, \hat{V}^p(\mathcal{A}_x))/K^p),$$

and this isomorphism η^p_K takes s_α to $s_{\alpha,\ell,x}$. At p, there is an isomorphism

$$V^{\vee}_{\mathbb{Z}_{p^r}} \xrightarrow{\sim} \mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^r})$$

taking s_α to $s_{\alpha,0,x}$.

These tensors make up somewhat for the lack of honest moduli structure in the sense that, together with the moduli data inherited from the Siegel modular variety, they distinguish points of our Shimura variety of Hodge type. In other words, the tensors distinguish points in the same fiber of the map $\overline{\mathscr{S}}_K(G,X) \to \overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^{\pm})$, as in the following result.

Proposition 2.5.3 ([Kis17, Cor 1.3.11]). *If $x, x' \in \overline{\mathscr{S}}_K(G,X)(\overline{\mathbb{F}}_p)$ lie over the same point of $\overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^{\pm})(\overline{\mathbb{F}}_p)$, then $x = x'$ if and only if $s_{\alpha,0,x} = s_{\alpha,0,x'}$ for all α.*

2.5.4 Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let v be a prime of E over p determined by this embedding. Denote the residue field by $k(v)$, and let $\overline{\mathscr{S}}_K(G,X) = \mathscr{S}_K(G,X) \otimes_{\mathcal{O}_{E,(p)}} k(v)$. We denote again by $\mathcal{A} \to \overline{\mathscr{S}}_K(G,X)$ the pullback of the abelian scheme $\mathcal{A} \to \mathscr{S}_K(G,X)$.

By [Lov17], the p-divisible group $\mathcal{A}[p^\infty]$ equipped with polarization and tensors on $\mathbb{D}(\mathcal{A}_x[p^\infty])$ gives rise to an isocrystal with G-structure over $\overline{\mathscr{S}}_K(G,X)$ in the sense of [RR96]. It therefore gives rise, as in 2.4.2, to a Newton stratification

$$\overline{\mathscr{S}}_K^{(\mathbf{b})}(G,X) = \{x \in \overline{\mathscr{S}}_K(G,X) : \mathbf{b}_x = \mathbf{b}\}$$

parametrized by classes $\mathbf{b} \in B(G)$. As in Siegel type, these are locally closed subsets, which we equip with the reduced subscheme structure. The $\mathbf{b}$-stratum is non-empty exactly when $\mathbf{b} \in B(G,\mu^{-1})$, where μ is a member of the conjugacy class of cocharacters arising from the Shimura datum (G,X) [KMS].

2.5.5 Fix a class $\mathbf{b} \in B(G,\mu^{-1})$ (corresponding to a non-empty Newton stratum) and let $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ be a p-divisible group with G-structure over $\overline{\mathbb{F}}_p$ of type $\mathbf{b}$, namely

- Σ a p-divisible group over $\overline{\mathbb{F}}_p$,
- λ_Σ a polarization of Σ, and
- $\{s_{\alpha,\Sigma}\} \subset \mathbb{D}(\Sigma)^\otimes$ a collection of tensors, such that
- there is an isomorphism $\mathbb{D}(\Sigma) \to V_{\mathcal{O}_L}$ preserving the pairings induced by λ_Σ and ψ, taking $s_{\alpha,\Sigma}$ to s_α, and taking the Frobenius on $D(\Sigma)$ to an endomorphism $b\sigma$ on $V_{\mathcal{O}_L}$ with $b \in G(\mathcal{O}_L)$ belonging to the fixed class $\mathbf{b} \in B(G)$.

Changing the isomorphism changes b by σ-conjugacy in $G(\mathcal{O}_L)$, so its class in $B(G)$ is well-defined.

As this data is required for the definition of Igusa varieties of Hodge type, we fix these choices of $\mathbf{b}$ and $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ for the remainder of the paper. We fix also an isomorphism $\mathbb{V}(\Sigma) \to V_L$, and therefore an element $b \in G(L)$ giving the Frobenius $b\sigma$ on $\mathbb{V}(\Sigma)$, which is a representative of our fixed class $\mathbf{b}$.

Define the central leaf corresponding to $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ by

$$C_{\Sigma,K} = \{x \in \overline{\mathscr{S}}_K^{(\mathbf{b})}(G,X) : (\mathcal{A}_x[p^\infty], \lambda_x, \{s_{\alpha,0,x}\}) \otimes_{k(x)} \overline{k(x)} \cong (\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\}) \otimes_{\overline{\mathbb{F}}_p} \overline{k(x)}\}.$$

This is a closed subset of the Newton stratum, and when equipped with the reduced subscheme structure, is smooth [Ham19, Prop 2.5].

2.5.6 We obtained the abelian scheme $\mathcal{A} \to \overline{\mathscr{S}}_K(G,X)$ by pulling back $\mathcal{A}' \to \overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^{\pm})$, and as a result all our constructions in Hodge type are compatible with those in Siegel type. To be precise, the map $G(L) \to \mathrm{GSp}(L)$ induces a map $B(G) \to B(\mathrm{GSp})$, and if we write $\mathbf{b}' \in B(\mathrm{GSp})$ for the image of our fixed class $\mathbf{b} \in B(G)$ then:

- the map $\overline{\mathscr{S}}_K(G,X) \to \overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^{\pm})$ sends $\overline{\mathscr{S}}_K^{(\mathbf{b})}(G,X)$ into $\overline{\mathscr{S}}_{K'}^{(\mathbf{b}')}(\mathrm{GSp}, S^{\pm})$;
- our p-divisible group with G-structure $(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\})$ of type $\mathbf{b}$ induces, by forgetting $\{s_{\alpha,\Sigma}\}$, a p-divisible group with GSp-structure (Σ, λ_Σ) of type $\mathbf{b}'$; and
- the map $\overline{\mathscr{S}}_K(G,X) \to \overline{\mathscr{S}}_{K'}(\mathrm{GSp}, S^{\pm})$ sends $C_{\Sigma,K}$ into $C'_{\Sigma,K'}$ (the central leaves taken with respect to the p-divisible groups just above).

Following [Ham19], we define the perfect infinite level Igusa variety of Hodge type

$$\mathscr{J}_{\Sigma,K} \subset (\mathrm{Ig}'_{\Sigma,K'} \times_{C'_{\Sigma,K'}} C_{\Sigma,K})^{(p^{-\infty})}$$

to be the locus of points where $j^*(s_{\alpha,0,x}) = s_{\alpha,\Sigma}$. That is, $\mathscr{J}_{\Sigma,K}$ parametrizes isomorphisms $\Sigma \otimes_{\overline{\mathbb{F}}_p} C_{\Sigma,K} \xrightarrow{\sim} \mathcal{A}[p^\infty]$ preserving polarizations (up to scaling) and tensors. Define (un-perfected) infinite level and finite level Igusa varieties of Hodge type by

$$\mathrm{Ig}_{\Sigma,K} = \mathrm{im}(\mathscr{J}_{\Sigma,K} \to \mathrm{Ig}'_{\Sigma,K'} \times_{C'_{\Sigma,K'}} C_{\Sigma,K}),$$
$$\mathrm{Ig}_{\Sigma,K,m} = \mathrm{im}(\mathscr{J}_{\Sigma,K} \to \mathrm{Ig}'_{\Sigma,K',m} \times_{C'_{\Sigma,K'}} C_{\Sigma,K}).$$

We will work primarily with the Igusa variety with infinite level at m and infinite level away from p, which we call

$$\mathrm{Ig}_\Sigma = \varprojlim_{K^p} \mathrm{Ig}_{\Sigma,K_pK^p}$$

(and analogously Ig'_Σ for the Siegel version). Since perfection does not change $\overline{\mathbb{F}}_p$-points nor étale cohomology, for these purposes it is essentially similar to work with the perfect Igusa variety. An important example is that the natural map $\mathscr{J}_{\Sigma,K} \to \mathscr{J}'_{\Sigma,K'}$ is a closed embedding [Ham19, Prop 4.10], so we can regard $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ as a subset of $\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$.

The Igusa variety Ig_Σ is not an honest moduli space, but we can nonetheless attach useful data to its points, which we will refer to as partial moduli data. A point $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ has associated data

$$(\mathcal{A}_x, \lambda_x, \eta^p, \{s_{\alpha,0,x}\}, j)$$

where $\mathcal{A}_x, \lambda_x, \eta^p, \{s_{\alpha,0,x}\}$ is the data attached to the image of x in $\overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$—on account of passing to infinite level away from p, the level structure η^p is now a full trivialization $V_{\mathbb{A}_f^p} \xrightarrow{\sim} \hat{V}^p(\mathcal{A}_x)$ sending s_α to $(s_{\alpha,\ell,x})_\ell$—and

$$j : \Sigma \xrightarrow{\sim} \mathcal{A}_x[p^\infty]$$

is the Igusa level structure attached to the image of x in the Siegel Igusa variety $\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$, an isomorphism of p-divisible groups over $\overline{\mathbb{F}}_p$ respecting polarizations up to scaling and sending $s_{\alpha,\Sigma}$ to $s_{\alpha,0,x}$.

Consider two sets of data to be equivalent if there is a prime-to-p isogeny between the abelian varieties sending one set of data to the other. With this equivalence, points are distinguished by their partial moduli data. Indeed, if two points of $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ agree on $(\mathcal{A}, \lambda, \eta^p)$ then they have the same image in $\overline{\mathscr{S}}_{K'_p}(\mathrm{GSp}, S^\pm)(\overline{\mathbb{F}}_p)$; if they agree furthermore on $\{s_{\alpha,0,x}\}$ then they have the same image in $\overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ by Proposition 2.5.3; and if they agree finally on j then they have the same image in $\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$. Together this implies the two points must be equal in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$.

2.5.7 The Igusa variety Ig_Σ (or system $\mathrm{Ig}_{\Sigma,K,m}$) receives an action of $G(\mathbb{A}_f^p)$ inherited from the Shimura variety, and a commuting action of the submonoid $S_b \subset J_b(\mathbb{Q}_p)$, which extends to an action of the full group $J_b(\mathbb{Q}_p)$ on the perfection and on étale cohomology. (Here b is a representative of our fixed class $\mathbf{b} \in B(G)$).

Let ξ be a finite-dimensional representation of G, and $\mathscr{L}_\xi$ the system of sheaves on $\mathrm{Ig}_{\Sigma,K,m}$ defined by ξ, as in [Kot92, §6]. Define

$$\begin{aligned} H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi) &= \varinjlim_{K^p,m} H_c^i(\mathrm{Ig}_{\Sigma,K,m}, \mathscr{L}_\xi), \\ H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi) &= \sum_i (-1)^i H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi), \end{aligned} \tag{2.5.8}$$

the latter as an element of $\mathrm{Groth}(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, where H_c^i denotes étale cohomology with compact supports. This is the representation that will be described by our eventual counting point formula.

We can describe the action of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ on the points $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ as follows. Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ with associated partial moduli data $(\mathcal{A}_x, \lambda_x, \eta^p, \{s_{\alpha,0,x}\}, j)$. Note that we can regard $J_b(\mathbb{Q}_p)$ as the group of self-quasi-isogenies of Σ.

The action of $G(\mathbb{A}_f^p)$ is inherited from the Shimura variety, and as there, it acts on the level structure η^p: for $g^p \in G(\mathbb{A}_f^p)$, the data associated to $x \cdot g^p$ is

$$(\mathcal{A}_x, \lambda_x, \eta^p \circ g^p, \{s_{\alpha,0,x}\}, j).$$

To describe the action of $g_p \in J_b(\mathbb{Q}_p)$, regard it as a quasi-isogeny $g_p : \Sigma \to \Sigma$, and choose $m \geq 0$ such that $p^m g_p^{-1} : \Sigma \to \Sigma$ is an isogeny. The Igusa level structure $j : \Sigma \xrightarrow{\sim} \mathcal{A}_x[p^\infty]$ allows us to transfer this to $\mathcal{A}_x$. The data associated to $x \cdot g_p$ is

$$(\mathcal{A}_x / j(\ker p^m g_p^{-1}), g_p^* \lambda_x, \eta^p, \{s_{\alpha,0,x}\}, g_p^* j)$$

where $g_p^* \lambda_x$ is the induced polarization; we can take the same level structure η^p because $\mathcal{A}_x$ is unchanged away from p, and the same tensors $\{s_{\alpha,0,x}\}$ because $J_b(\mathbb{Q}_p)$, being a subgroup of $G(L)$, preserves tensors; and

$$g_p^* j : \Sigma \xrightarrow{\sim} \mathcal{A}_{x \cdot g_p}[p^\infty] = \mathcal{A}_x[p^\infty] / j(\ker p^m g_p^{-1})$$

is the unique map making the following diagram commute.

$$\begin{array}{ccc} \Sigma & \xrightarrow{j} & \mathcal{A}_x[p^\infty] \\ {\scriptstyle p^m g_p^{-1}} \downarrow & & \downarrow \\ \Sigma & \xrightarrow[g_p^* j]{} & \mathcal{A}_x[p^\infty] / j(\ker p^m g_p^{-1}) \end{array}$$

Note that the choice of m does not matter because multiplication by p^k induces an isomorphism $A / \ker p^k \to A$, and this gives an equivalence between moduli data for different choices of m.

2.5.9 There is an alternative partial moduli description which admits a simpler description of the group actions, but has the downside that it makes the map to $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$ more opaque. By [Shi09, Lem 7.1], we have the following moduli description of the Siegel Igusa variety:

$$\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p) = \{(A, \lambda, \eta^p, j)\} / \sim$$

where

- A is an abelian variety over $\overline{\mathbb{F}}_p$,
- λ is a polarization of A,
- $\eta^p : V_{\mathbb{A}_f^p} \xrightarrow{\sim} \hat{V}^p(A)$ is an isomorphism preserving the pairings induced by *psi* and λ up to scaling,
- $j : \Sigma \to A[p^\infty]$ is a quasi-isogeny preserving polarizations up to scaling, and
- two tuples are equivalent if there is an isogeny $A_1 \to A_2$ sending λ_1 to a scalar multiple of λ_2, and sending η_1^p to η_2^p and j_1 to j_2. (It is equivalent to replace "isogeny" here with "quasi-isogeny").

Note the difference that we allow j to be a quasi-isogeny rather than an isomorphism, and equivalence requires only an isogeny $A_1 \to A_2$, rather than a prime-to-p isogeny.

Under this moduli description, $\mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$ has a right action of $\mathrm{GSp}(\mathbb{A}_f^p) \times J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$ (where we write b again for the image of b in $\mathrm{GSp}(L)$) described by

$$(g^p, g_p) : (A, \lambda, \eta^p, j) \mapsto (A, \lambda, \eta^p \circ g^p, j \circ g_p).$$

As noted in 2.5.6, we can regard $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \subset \mathrm{Ig}'_\Sigma(\overline{\mathbb{F}}_p)$, and furthermore this is compatible with the actions of $G(\mathbb{A}_f^p) \times J_b^G(\mathbb{Q}_p) \subset \mathrm{GSp}(\mathbb{A}_f^p) \times J_b^{\mathrm{GSp}}(\mathbb{Q}_p)$. Thus each point of $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ can be associated data (A, λ, η^p, j) as above, with distinct points having distinct data, and we can write the action of $G(\mathbb{A}_f^p) \times J_b^G(\mathbb{Q}_p)$ in a precisely similar way.

This simpler description of the action will be useful, but note also that under this partial moduli description the data associated to the image our point in $\overline{\mathscr{S}}_{K'_p}(\mathrm{GSp}, S^\pm)$ may not be (A, λ, η^p)—simply forgetting j—as we have allowed p-power isogenies of A in our notion of equivalence (whereas the data on the Siegel modular variety are equivalent only under prime-to-p isogenies).

2.6 Galois Gerbs

In this section we review some of the theory of Galois gerbs that will be important for our work, as well as proving some technical lemmas. We refer to §3 of [Kis17] and §2 of [KSZ21] for details omitted here.

2.6.1 A k'/k-*Galois gerb* is a linear algebraic group G over k' and an extension of topological groups (giving $G(k')$ the discrete topology)

$$1 \to G(k') \to \mathfrak{G} \to \mathrm{Gal}(k'/k) \to 1$$

satisfying certain technical conditions [Kis17, 3.1.1]. We will use the name of the extension $\mathfrak{G}$ also to refer to the whole data, and we say that $\mathfrak{G}^\Delta = G$ is the *kernel* of $\mathfrak{G}$. We sometimes refer to a $\overline{k}/k$-Galois gerb as simply a Galois gerb over k.

A k'/k-Galois gerb $\mathfrak{G}$ induces a $\overline{k}/k$-Galois gerb by pulling back by $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(k'/k)$ and pushing out by $G(k') \to G(\overline{k})$. Similarly, for any place v of $\mathbb{Q}$, a Galois gerb $\mathfrak{G}$ over $\mathbb{Q}$ induces a Galois gerb $\mathfrak{G}(v)$ over $\mathbb{Q}_v$ by pulling back by $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and pushing out by $G(\overline{\mathbb{Q}}) \to G(\overline{\mathbb{Q}}_v)$.

An important example will be the *neutral* k'/k-*Galois gerb* attached to a group G over k, defined to be the semi-direct product $\mathfrak{G}_G = G(k') \rtimes \mathrm{Gal}(k'/k)$, where the action of $\mathrm{Gal}(k'/k)$ on $G(k')$ is given by the k-structure on G.

A morphism of k'/k-Galois gerbs $f : \mathfrak{G}_1 \to \mathfrak{G}_2$ is a continuous group homomorphism which fits into a commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1(k') & \longrightarrow & \mathfrak{G}_1 & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f^{\Delta}(k')} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & G_2(k') & \longrightarrow & \mathfrak{G}_2 & \longrightarrow & \mathrm{Gal}(k'/k) & \longrightarrow & 1 \end{array}$$

where $f^{\Delta}(k')$ is induced by a map of algebraic groups $f^{\Delta} : G_1 \to G_2$. If $\mathfrak{G}_1, \mathfrak{G}_2$ are Galois gerbs over $\mathbb{Q}$, then f induces a morphism $f(v) : \mathfrak{G}_1(v) \to \mathfrak{G}_2(v)$ of Galois gerbs over $\mathbb{Q}_v$. Two morphism $\mathfrak{G}_1 \to \mathfrak{G}_2$ are *conjugate* if they are related by conjugation by an element of $G_2(k')$. We denote the conjugacy class of a morphism f by $[f]$.

Let $f : \mathfrak{G}_1 \to \mathfrak{G}_2$ be a morphism. If R is a k-algebra, we can consider the pushouts $\mathfrak{G}_{1,R}, \mathfrak{G}_{2,R}$ of $\mathfrak{G}_1, \mathfrak{G}_2$ by the map $G_1(k') \to G_1(k' \otimes_k R)$, and f induces a map $\mathfrak{G}_{1,R} \to \mathfrak{G}_{2,R}$; we define the automorphism group of the morphism f by

$$I_f(R) = \{g \in G_2(k' \otimes_k R) : \mathrm{Int}(g) \circ f = f\}.$$

This defines a group scheme I_f over k.

In the case that $\mathfrak{G}_1$ is any Galois gerb and $\mathfrak{G}_2 = \mathfrak{G}_G$ is the neutral Galois gerb attached to a linear algebraic group G, we have the following lemma.

Lemma 2.6.2 ([Kis17, Lem 3.1.2]). *Let $f : \mathfrak{G}_1 \to \mathfrak{G}_G$ be a map of k'/k-Galois gerbs.*

- *The map $I_{f,k'} \to G_{k'}$ given by*

  $$I_{f,k'}(R) \longrightarrow G(k' \otimes_k R) \to G(R)$$

  (where R is a k'-algebra, and $k' \otimes_k R \to R$ is the multiplication map) identifies $I_{f,k'}$ with the centralizer $Z_G(f^{\Delta})$ in $G_{k'}$.

- *The set of morphisms $f' : \mathfrak{G}_1 \to \mathfrak{G}_G$ with $f'^{\Delta} = f^{\Delta}$ is in bijection with $Z^1(\mathrm{Gal}(k'/k), I_f(k'))$, via the map sending $e \in Z^1(\mathrm{Gal}(k'/k), I_f(k'))$ to the morphism ef defined such that, if $f(q) = g \rtimes \rho$, we have $ef(q) = e_\rho g \rtimes \rho$. Furthermore, ef is conjugate to $e'f$ exactly when e is cohomologous to e'.*

2.6.3 There is a distinguished Galois gerb over $\mathbb{Q}$ called the *quasi-motivic Galois gerb*, and denoted $\mathfrak{Q}$, which plays a central role in point counting. We refer to [Kis17, 3.1] for full details, but we review here the essential properties we will need.

For $L/\mathbb{Q}$ a finite Galois extension, define

$$Q^L = (Res_{L(\infty)/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{L(p)/\mathbb{Q}}\,\mathbb{G}_m)/\mathbb{G}_m,$$

where the action of $\mathbb{G}_m$ is the diagonal action, and $L(\infty) = L \cap \mathbb{R}$ and $L(p) = L \cap \mathbb{Q}_p$. This group is equipped with cocharacters $\nu(p)^L$ over $\mathbb{Q}_p$ and $\nu(\infty)^L$ over $\mathbb{R}$ defined by

$$\nu(v)^L : \mathbb{G}_m \to \mathrm{Res}_{L(v)/\mathbb{Q}} \mathbb{G}_m \to Q^L.$$

For L'/L Galois there is a natural map $Q^{L'} \to Q^L$, and the limit is a pro-torus $Q = \varprojlim_L Q^L$ over $\mathbb{Q}$ equipped with a fractional cocharacter $\nu(p) : \mathbb{D} \to Q$ over $\mathbb{Q}_p$ and cocharacter $\nu(\infty) : \mathbb{G}_m \to Q$ over $\mathbb{R}$. The kernel of the quasi-motivic Galois gerb is this pro-torus $\mathfrak{Q}^\Delta = Q$.

Intuitively, the quasi-motivic Galois gerb $\mathfrak{Q}$ should be thought of as the fundamental group of the category of motives over $\overline{\mathbb{F}}_p$, and a morphism $\mathfrak{Q} \to \mathfrak{G}_G$ as a representation that corresponds to a motive with G-structure. Since the motive of an abelian variety is essentially the isogeny class of that abelian variety, this gives a motivation for the role played by morphisms $\mathfrak{Q} \to \mathfrak{G}_G$ in parametrizing "isogeny classes" in our later point-counting work.

In line with this intuition, the quasi-motivic Galois gerb $\mathfrak{Q}$ has a v-adic realization for any place v of $\mathbb{Q}$ in the form of a morphism $\zeta_v : \mathfrak{G}_v \to \mathfrak{Q}(v)$ from a distinguished Galois gerb over $\mathbb{Q}_v$.

At $\ell \neq p, \infty$

$$\mathfrak{G}_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$$

is the trivial Galois gerb.

At p, we have

$$\mathfrak{G}_p = \varprojlim_L \mathfrak{G}_p^L$$

where L runs over finite Galois extensions of $\mathbb{Q}_p$, and $\mathfrak{G}_p^L$ is the $L/\mathbb{Q}_p$-gerb (induced to $\overline{\mathbb{Q}}_p$) with kernel $\mathfrak{G}_p^{L,\Delta} = \mathbb{G}_m$ given by the fundamental class in $H^2(\mathrm{Gal}(L/\mathbb{Q}_p), L^\times)$. The kernel of $\mathfrak{G}_p$ is $\mathfrak{G}_p^\Delta = \mathbb{D}$, the pro-torus with character group $\mathbb{Q}$.

At ∞,

$$1 \to \mathbb{C}^\times \to \mathfrak{G}_\infty \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$$

is the extension corresponding to the fundamental class in $H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$.

At p and ∞ we have $\zeta_p^\Delta = \nu(p)$ and $\zeta_\infty^\Delta = \nu(\infty)$; these cocharacters will play a role in some later arguments.

The quasi-motivic Galois gerb is also equipped with a distinguished morphism $\psi : \mathfrak{Q} \to \mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}} \mathbb{G}_m}$ which allows us to construct morphisms to neutral Gerbs from cocharacters in the following way. Let T be a torus over $\mathbb{Q}$ and μ a cocharacter of T defined over a Galois extension $L/\mathbb{Q}$. Then μ induces a map $\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m \to T$ which further induces a morphism of Galois gerbs $\mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}}} \to \mathfrak{G}_T$. We define a morphism $\psi_\mu : \mathfrak{Q} \to \mathfrak{G}_T$ by the composition

$$\psi_\mu : \mathfrak{Q} \xrightarrow{\psi} \mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}}} \longrightarrow \mathfrak{G}_{\mathrm{Res}_{L/\mathbb{Q}}} \longrightarrow \mathfrak{G}_T.$$

2.6.4 The realization at p will play an especially important role, and for this we need to introduce the notion of unramified morphisms from $\mathfrak{G}_p$.

Recall that $\mathfrak{G}_p$ was defined as a limit of Galois gerbs $\mathfrak{G}_p^L$, where L runs over finite Galois extensions of $\mathbb{Q}_p$. If we restrict L to run unramified extensions, we can define a $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-Galois gerb $\mathfrak{D}$ with kernel $\mathfrak{D}^\Delta = \mathbb{D}$, which becomes $\mathfrak{Q}_p$ when induced to $\overline{\mathbb{Q}}_p$. Writing $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ for the Frobenius, there is a distinguished element $d_\sigma \in \mathfrak{D}$ lying over σ and such that d_σ^n maps to $p^{-1} \in \mathbb{G}_m = \mathfrak{G}_p^{\mathbb{Q}_{p^n},\Delta}$ under the projection to $\mathfrak{G}_p^{\mathbb{Q}_{p^n}}$.

Definition 2.6.5. A morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$ is *unramified* if it is induced by a morphism $\theta^{\mathrm{ur}} : \mathfrak{D} \to \mathfrak{G}_G^{\mathrm{ur}}$, where $\mathfrak{G}_G^{\mathrm{ur}}$ is the neutral $\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p$-gerb attached to G. For θ an unramified morphism, we define an element $b_\theta \in G(\mathbb{Q}_p^{\mathrm{ur}})$ by $\theta^{\mathrm{ur}}(d_\sigma) = b_\theta \rtimes \sigma$.

2.6.6 If G is connected, every morphism $f : \mathfrak{G}_p \to \mathfrak{G}_G$ is conjugate to an unramified morphism [KSZ21, Lem 2.6.3 (ii)]. If θ and θ' are unramified morphisms conjugate to f, then b_θ and $b_{\theta'}$ are related by σ-conjugacy in $G(\mathbb{Q}_p^{\mathrm{ur}})$, so we can associate to f a well-defined class $[b_\theta] \in B(G)$. Under our intuition that a morphism $\mathfrak{Q} \to \mathfrak{G}_G$ corresponds to a motive, the morphism $\mathfrak{G}_p \to \mathfrak{G}_G$ is the p-adic realization, giving an isocrystal classified by $[b_\theta]$. The following lemma gives some concrete illustration of this intuition.

Lemma 2.6.7 ([KSZ21, Prop 2.6.5])**.** *Let G be a connected linear algebraic $\mathbb{Q}_p$-group, $\theta : \mathfrak{G}_p \to \mathfrak{G}_G$ an unramified morphism, and ν the fractional cocharacter $\theta^{\mathrm{ur},\Delta} : \mathbb{D}_{\mathbb{Q}_p^{\mathrm{ur}}} \to G_{\mathbb{Q}_p^{\mathrm{ur}}}$. Then*

- *$\nu = -\nu_{b_\theta}$, where ν_{b_θ} is the slope homomorphism of b_θ, and*
- *there are natural $\mathbb{Q}_p$-isomorphisms $J_{b_\theta} \cong I_{\theta^{\mathrm{ur}}} \cong I_\theta$.*

2.6.8 The morphisms $\mathfrak{Q} \to \mathfrak{G}_G$ that will be used in our point-counting are required to satisfy an admissibility condition. For $\ell \neq p, \infty$, let $\xi_\ell : \mathfrak{G}_\ell \to \mathfrak{G}_G(\ell)$ be the map sending $\rho \mapsto 1 \rtimes \rho$.

Definition 2.6.9. A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is *admissible* if

- for $\ell \neq p, \infty$, the morphism $\phi(\ell) \circ \zeta_\ell : \mathfrak{G}_\ell \to \mathfrak{G}_G(\ell)$ is conjugate to ξ_ℓ;
- at p, the morphism $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate to an unramified morphism θ such that $b_\theta \in G(\mathbb{Z}_p^{\mathrm{ur}})\mu(p)^{-1}G(\mathbb{Z}_p^{\mathrm{ur}})$;

as well as satisfying a global condition [Kis17, 3.3.6 (1)].

An important example of admissible morphisms is given by the morphisms ψ_μ constructed in 2.6.3. Namely, if $T \subset G$ is a torus and $\mu \in X_*(T)$, we have an admissible morphism for G defined by

$$i \circ \psi_\mu : \mathfrak{Q} \xrightarrow{\psi_\mu} \mathfrak{G}_T \xrightarrow{i} \mathfrak{G}_G.$$

2.6.10 As in Lemma 2.6.2 we can twist a morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ by a cocycle $e \in Z^1(\mathbb{Q}, I_\phi)$.

Define

$$Ш^S_G(\mathbb{Q}, I_\phi) \subset H^1(\mathbb{Q}, I_\phi)$$

to be the subset of classes which are trivial in $H^1(\mathbb{Q}, G)$ as well as $H^1(\mathbb{Q}_v, I_\phi)$ for $v \in S$. For our purposes S will be $\{\infty\}$ or $\{p, \infty\}$ or $\{\text{all places of } \mathbb{Q}\}$. In the case that S is all places of $\mathbb{Q}$, we also write $Ш_G(\mathbb{Q}, I_\phi)$ for $Ш^S_G(\mathbb{Q}, I_\phi)$. See [KSZ21, 1.4.6] for more details.

The following proposition characterizes when the twist of an admissible morphism is again admissible.

Proposition 2.6.11 ([KSZ21, Prop 3.1.13])**.** *If ϕ is an admissible morphism and $e \in Z^1(\mathbb{Q}, I_\phi)$, then $e\phi$ is admissible exactly when e lies in $Ш^\infty_G(\mathbb{Q}, I_\phi)$.*

2.6.12 For an admissible morphism ϕ we can define a set

$$X^p(\phi) = \{x = (x_\ell) \in G(\overline{\mathbb{A}}^p_f) : \mathrm{Int}(x_\ell) \circ \xi_\ell = \phi(\ell) \circ \zeta_\ell\}.$$

It is non-empty by the admissible condition for $\ell \neq p, \infty$, and furthermore is a $G(\mathbb{A}^p_f)$-torsor under the natural right action (note that $I_{\xi_\ell}(\mathbb{Q}_\ell) = G(\mathbb{Q}_\ell)$).

Let $x \in X^p(\phi)$, and define a cocycle $\zeta^{p,\infty}_\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G(\overline{\mathbb{A}}^p_f)$ by $\rho \mapsto x\rho(x)^{-1}$. This does not depend on the choice of x, because any other choice x' is related by $x' = xg$ for some $g \in G(\mathbb{A}^p_f)$, and since g is rational we have $x'\rho(x')^{-1} = xg\rho(xg)^{-1} = xgg^{-1}\rho(x)^{-1} = x\rho(x)^{-1}$.

Define cocycles $\zeta_{\phi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \to G(\overline{\mathbb{Q}}_\ell)$ by

$$\zeta_{\phi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\zeta^{p,\infty}_\phi} G(\overline{\mathbb{A}}^p_f) \to G(\overline{\mathbb{Q}}_\ell),$$

where the map $G(\overline{\mathbb{A}}^p_f) \to G(\overline{\mathbb{Q}}_\ell)$ is induced by the map $\overline{\mathbb{A}}^p_f \to \overline{\mathbb{Q}}_\ell$ sending a point to its ℓ-component. These cocycles record the component of ϕ at ℓ in the sense that $(\phi(\ell) \circ \zeta_\ell)(\rho) = \zeta_{\phi,\ell}(\rho) \rtimes \rho$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

3 Langlands-Rapoport Conjecture for Igusa Varieties of Hodge Type

In broad view, our strategy for deriving a point counting formula for Igusa varieties of Hodge type is to use Fujiwara's formula (packaged in the definition of acceptable functions 4.1.6) to express the trace of an operator on $H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ in terms of fixed points of a correspondence on Ig_Σ. To work with these fixed points, we need a good description of the set $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ and its $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-action. The goal of this section is to obtain such a description, namely Theorem 3.6.2, analogous to the description of $\overline{\mathbb{F}}_p$-points of a Shimura variety given by the Langlands-Rapoport conjecture. Indeed, our work relies heavily on the work of Kisin [Kis17] and Kisin-Shin-Zhu [KSZ21] on the Langlands-Rapoport conjecture and subsequent point-counting for Shimura varieties.

The Langlands-Rapoport description can be understood as breaking the space down into "isogeny classes" (in reference to the moduli data). The description then breaks down into two separate pieces: describing the points in an individual isogeny class; and describing the set of isogeny classes. These "descriptions" are written in terms of Galois gerbs.

The connection between Galois gerbs and isogeny classes is made by (refined) Kottwitz triples, which can be associated to both Galois gerbs and isogeny classes, and which capture much of the information on both sides. The main work that goes into the "descriptions" above is connecting both sides to Kottwitz triples.

In §3.1 we recall the description of isogeny classes on Shimura varieties of Hodge type. In §3.2 we define isogeny classes on Igusa varieties (Definition 3.2.2) and relate them to isogeny classes on Shimura varieties (Proposition 3.2.5). In §3.3 we introduce the Galois gerbs that will be used in the parametrization. In §3.4 we recall the link between Galois gerbs and isogeny classes given by Kottwitz triples and special point data. In §3.6 these ingredients are all combined to state and prove the main Theorem 3.6.2.

3.1 Isogeny Classes on Shimura Varieties of Hodge Type

We begin by recalling from [Kis17] the description of isogeny classes on Shimura varieties of Hodge type, which will be important for our description of isogeny classes on Igusa varieties.

3.1.1 Following [Kis17, 1.4.1] we define a cocharacter v of G and an element $b \in G(L)$. These are needed to define the affine Deligne-Lusztig variety $X_v(b)$, which records the p-part of an isogeny class on the Shimura variety.

Let $x \in \overline{\mathscr{S}}_K(G,X)(\mathbb{F}_{p^r})$ for r divisible by the residue degree of v (our fixed prime of E lying over p as in 2.5.4). As in 2.5.2, there is an isomorphism

$$V^\vee_{\mathbb{Z}_{p^r}} \xrightarrow{\sim} \mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^r})$$

taking s_α to $s_{\alpha,0,x}$. This allows us to identify

$$G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{p^r} \xrightarrow{\sim} G_{0,x} \subset \mathrm{GL}(\mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^r})), \tag{3.1.2}$$

where $G_{0,x}$ is the subgroup fixing the tensors $s_{\alpha,0,x}$. Under this identification, the action of Frobenius is given by $b\sigma$, where $b \in G(\mathbb{Q}_{p^r})$ and σ is the lift of Frobenius on $\mathbb{Z}_{p^r}$. The element b is well-defined up to σ-conjugacy by $G(\mathbb{Z}_{p^r})$.

Fix a Borel B and maximal torus T in $G_{\mathbb{Z}_{(p)}}$, let $\mu \in X_*(T)$ be the dominant cocharacter in the conjugacy class of μ_h for $h \in X$, and let $v = \sigma(\mu^{-1})$. Then we have $b \in G(\mathbb{Z}_{p^r})v(p)G(\mathbb{Z}_{p^r})$.

Define the *affine Deligne-Lusztig variety*

$$X_v(b) = \{g \in G(L)/G(\mathcal{O}_L) : g^{-1}b\sigma(g) \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)\}.$$

We consider $X_v(b)$ as a set, and equip it with a Frobenius operator

$$\Phi(g) = (b\sigma)^r g = b \cdot \sigma(b) \cdots \sigma^{r-1}(b) \cdot \sigma^r(g).$$

Note that it also carries an action of $J_b(\mathbb{Q}_p)$ by left multiplication, as $J_b(\mathbb{Q}_p)$ is the group of elements of $G(L)$ which σ-centralize b.

Intuitively, choosing $g \in G(L)/G(\mathcal{O}_L)$ corresponds to choosing a lattice with "G-structure" in V_L, and requiring $g^{-1}b\sigma(g) \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)$ ensures that this lattice is stable under Frobenius and satisfies the axioms of a Dieudonné module. This Dieudonné module corresponds to a p-divisible group with G-structure which is isogenous to $\mathcal{A}_x[p^\infty]$, as they have the same isocrystal. Thus points of $X_v(b)$ correspond roughly to isomorphism classes of p-divisible groups in a fixed isogeny class, and this explains their role in defining the p-part of our isogeny classes on Shimura varieties.

3.1.3 Following [Kis17, 1.4.2] we define a map $X_v(b) \to \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ as follows. Choose a base point $x \in \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$, with associated p-divisible group $\mathcal{A}_x[p^\infty]$. For $g \in X_v(b)$, the lattice $g \cdot \mathbb{D}(\mathcal{A}_x[p^\infty]) \subset \mathbb{V}(\mathcal{A}_x[p^\infty])$ is again a Dieudonné module, and corresponds to a p-divisible group $\mathscr{G}_{gx}$ equipped with a quasi-isogeny $\mathcal{A}_x[p^\infty] \to \mathscr{G}_{gx}$.

Let $\mathcal{A}_{gx}$ be the corresponding abelian variety equipped with the polarization and level structure induced from $\mathcal{A}_x$. Sending $g \mapsto \mathcal{A}_{gx}$ with polarization and level structure defines a map $X_v(b) \to \overline{\mathscr{S}}_{K'_p}(\mathrm{GSp}, S^\pm)(\overline{\mathbb{F}}_p)$. By [Kis17, Prop 1.4.4] this map has a unique lift to a map $i_x : X_v(b) \to \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ satisfying $s_{\alpha,0,x} = s_{\alpha,0,i_x(g)} \in \mathbb{D}(\mathcal{A}_{gx}[p^\infty])$. Extending by the action of $G(\mathbb{A}_f^p)$, we get a map

$$i_x : G(\mathbb{A}_f^p) \times X_v(b) \longrightarrow \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p) \tag{3.1.4}$$

which is equivariant for the action of $G(\mathbb{A}_f^p)$ and intertwines the action of Φ on $X_v(b)$ with the action of geometric p^r-Frobenius on $\overline{\mathscr{S}}_{K_p}(G,X)$.

Definition 3.1.5. For $x \in \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$, the *isogeny class of* x, denoted $\mathscr{I}_x^{\mathrm{Sh}}$, is the image of the map (3.1.4).

The set $G(\mathbb{A}_f^p) \times X_v(b)$ is still not quite right to parametrize an isogeny class, in the sense that the map (3.1.4) is not injective. We can describe the fibers by the action of a certain automorphism group, and quotient to get an injective map and a correct parametrization of the isogeny class.

3.1.6 Define $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ to be the algebraic group over $\mathbb{Q}$ with points

$$\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(R) = (\mathrm{End}_{\mathbb{Q}}(\mathcal{A}_x) \otimes_{\mathbb{Q}} R)^\times,$$

and define $I_x \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ to be the subgroup preserving the polarization of $\mathcal{A}_x$ up to scaling and fixing the tensors $s_{\alpha,\ell,x}$ $(\ell \neq p)$ and $s_{\alpha,0,x}$.

The level structure away from p

$$\eta^p : V_{\mathbb{A}_f^p} \xrightarrow{\sim} \hat{V}^p(\mathcal{A}_x)$$

identifies the tensors s_α and $(s_{\alpha,\ell,x})_{\ell\neq p}$, and therefore identifies $G(\mathbb{A}_f^p)$ with the subgroup of $\mathrm{GL}(\hat{V}^p(\mathcal{A}_x))$ fixing $(s_{\alpha,\ell,x})_{\ell\neq p}$. Thus the embedding $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)(\mathbb{Q}) \hookrightarrow \mathrm{GL}(\hat{V}^p(\mathcal{A}_x))$ induces an embedding

$$I_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p),$$

canonical up to conjugation by $G(\mathbb{A}_f^p)$.

Similarly, our above identification (3.1.2) allows us to identify $J_b(\mathbb{Q}_p)$ with the subgroup of $\mathrm{GL}(\mathbb{V}(\mathcal{A}_x[p^\infty]))$ fixing the tensors $s_{\alpha,0,x}$ and commuting with the Frobenius. Thus the embedding $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A})(\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathbb{V}(\mathcal{A}_x[p^\infty]))$ induces an embedding

$$I_x(\mathbb{Q}) \hookrightarrow J_b(\mathbb{Q}_p),$$

canonical up to conjugation by $J_b(\mathbb{Q}_p)$.

Thus we have an embedding

$$I_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p),$$

canonical up to conjugation. We fix such a choice of embedding.

Through this embedding, $I_x(\mathbb{Q})$ acts on the set $G(\mathbb{A}_f^p) \times X_v(b)$. By [Kis17, Prop 2.1.3], the map (3.1.4) induces an injective map

$$i_x : I_x(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times X_v(b) \hookrightarrow \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p). \tag{3.1.7}$$

Thus the isogeny class of a point $x \in \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ is parametrized by the set $I_x(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times X_v(b)$.

We can also give a description of isogeny classes in terms of the partial moduli structure, which (in addition to being useful) gives a very plain relation to isogenies of the moduli data.

Proposition 3.1.8 ([Kis17, Prop 1.4.15])**.** *Two points $x, x' \in \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ lie in the same isogeny class exactly when there is a quasi-isogeny $\mathcal{A}_x \to \mathcal{A}_{x'}$ preserving polarizations up to scaling and such that the induced maps $\mathbb{D}(\mathcal{A}_{x'}[p^\infty]) \to \mathbb{D}(\mathcal{A}_x[p^\infty])$ and $\hat{V}^p(A_x) \to \hat{V}^p(\mathcal{A}_{x'})$ send $s_{\alpha,0,x'}$ to $s_{\alpha,0,x}$ and $s_{\alpha,\ell,x}$ to $s_{\alpha,\ell,x'}$.*

3.2 Isogeny Classes on Igusa Varieties of Hodge Type

We now define isogeny classes on Igusa varieties, and relate them to isogeny classes on Shimura varieties. The main change happens at p, where the affine Deligne-Lusztig variety $X_v(b)$ is replaced by the group $J_b(\mathbb{Q}_p)$. Recall that

$$X_v(b) = \{g \in G(L)/G(\mathcal{O}_L) : gb\sigma(g)^{-1} \in G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)\},$$
$$J_b(\mathbb{Q}_p) = \{g \in G(L) : gb\sigma(g)^{-1} = b\}.$$

We can interpret this change in relation to the intuition for affine Deligne-Lusztig variety mentioned in 3.1.1. The stricter requirement that $gb\sigma(g)^{-1} = b$ corresponds to the fact that our Igusa variety lives over a central leaf which fixes the isomorphism class of the p-divisible group, as opposed to allowing it to vary in its isogeny class. The change from $G(L)/G(\mathcal{O}_L)$ to $G(L)$ corresponds to the fact that the Igusa variety carries level structure in the form of a trivialization of the p-divisible group, so we care not only about the isomorphism class, but even about the choice of isomorphism.

With this change, the parametrization will look quite similar.

Recall from 3.1.6 the group I_x and embedding $I_x(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. We begin by giving the parametrization (and definition) of an isogeny class in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$.

Lemma 3.2.1. *For $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, the map*

$$\begin{aligned} i_x : I_x(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p) &\to \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \\ (g^p, g_p) &\mapsto x \cdot (g^p, g_p) \end{aligned}$$

is well-defined and injective.

Proof. To show that this map is well-defined and injective is to show that $I_x(\mathbb{Q})$ is the stabilizer of x under the action of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. We use the partial moduli structure described in 2.5.9, as it provides a simpler description of the group action. Let (A, λ, η^p, j) be the data associated to x. Then the data associated to $x \cdot (g^p, g_p)$ is $(A, \lambda, \eta^p \circ g^p, j \circ g_p)$.

If $x = x \cdot (g^p, g_p)$, then by the partial moduli interpretation there is a quasi-isogeny $\theta : A \to A$ preserving λ up to $\mathbb{Q}^\times$-scaling and sending η^p to $\eta^p \circ g^p$ and sending j to $j \circ g_p$. This implies that θ acts as g^p on $\hat{V}^p(A)$, and therefore preserves the tensors $s_{\alpha,\ell,x}$; also θ acts as g_p on $\mathbb{V}(\mathcal{A}[p^\infty])$, and therefore preserves the tensors $s_{\alpha,0,x}$. Thus we can regard θ as an element of $I_x(\mathbb{Q})$, which is identified with (g^p, g_p) under our embedding; i.e. $(g^p, g_p) = \theta \in I_x(\mathbb{Q})$.

Conversely, suppose that $(g^p, g_p) = \theta \in I_x(\mathbb{Q})$. Then $\theta : A \to A$ is a quasi-isogeny preserving the polarization up to $\mathbb{Q}^\times$ and sending η^p to $\eta^p \circ g^p$ and sending j to $j \circ g_p$. Thus θ gives an equivalence between (A, λ, η^p, j) and $(A, \lambda, \eta^p \circ g^p, j \circ g_p)$ in the moduli description, showing $x = x \cdot (g^p, g_p)$. □

Definition 3.2.2. For $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, the *isogeny class of* x, denoted $\mathscr{I}_x^{\mathrm{Ig}}$, is the image of the map of 3.2.1. We may also speak of an *isogeny class* $\mathscr{I}^{\mathrm{Ig}}$ if we do not specify x.

The next two lemmas relate isogeny classes on the Igusa variety and the Shimura variety. The essential relationship is stated in Proposition 3.2.5.

Lemma 3.2.3. *Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, and $x' \in \overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$ the image of x under the natural map. The isogeny class maps for x and x' fit into a pullback diagram as below, where the left vertical map is defined by $(g^p, g_p) \mapsto (g^p, g_p)$.*

$$\begin{array}{ccc} I_x(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p) & \xrightarrow{\ i_x\ } & \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \\ \downarrow & & \downarrow \\ I_{x'}(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times X_v(b) & \xrightarrow[\ i_{x'}\ ]{} & \overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p) \end{array}$$

That is, an isogeny class on the Igusa variety is the preimage of an isogeny class on the Shimura variety.

Proof. Recall from 3.1.1 that we have chosen b so that $b \in G(\mathbb{Z}_{p^r})v(p)G(\mathbb{Z}_{p^r})$, which ensures that the coset of $1 \in G(L)$ is in $X_v(b)$. By $(g^p, g_p) \mapsto (g^p, g_p)$ we mean mapping $G(\mathbb{A}_f^p) \to G(\mathbb{A}_f^p)$ by the identity, and mapping $J_b(\mathbb{Q}_p) \to X_v(b)$ by sending the identity to the identity coset and extending by the action of $J_b(\mathbb{Q}_p)$ on $X_v(b)$ by left multiplication. This map descends to the quotients by $I_x(\mathbb{Q})$ and $I_{x'}(\mathbb{Q})$ because in fact $I_x = I_{x'}$ (since x and x' have the same abelian variety with G-structure in their associated data) and the action of $I_{x'}(\mathbb{Q})$ on $G(\mathbb{A}_f^p) \times X_v(b)$ is defined via its embedding in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$.

Next we show that this diagram commutes. For this we can ignore the quotients by $I_x(\mathbb{Q})$. The whole diagram is $G(\mathbb{A}_f^p)$-equivariant, so it suffices to check commutativity for elements of $J_b(\mathbb{Q}_p)$. Let $(1, g_p) \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. We use the partial moduli description of 2.5.6, because it provides a simpler description of the map to the Shimura variety. Write $(A_x, \lambda, \eta^p, \{s_{\alpha,0,x}\}, j)$ for the data at x. Then the data at $x \cdot g_p$, i.e. the image of $(1, g_p)$ in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, is

$$(A_x/j(\ker p^m g_p^{-1}), g_p^*\lambda, \{s_{\alpha,0,x}\}, \eta^p, g_p^* j),$$

and the data at the image in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$ (following the right vertical arrow of the diagram) is

$$(A_x/j(\ker p^m g_p^{-1}), g_p^*\lambda, \{s_{\alpha,0,x}\}, \eta^p).$$

Going the other way around the diagram, the image in $G(\mathbb{A}_f^p) \times X_v(b)$ is $(1, g_p)$. The image in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$ (following the bottom horizontal arrow) is defined as in 3.1.3 by taking the p-divisible group $\mathscr{G}_{g_p x}$ associated to the Dieudonné module $g_p \cdot \mathbb{D}(\mathcal{A}_x[p^\infty])$, with quasi-isogeny $\mathcal{A}_x[p^\infty] \to \mathscr{G}_{g_p x}$ induced by the isomorphism $g_p^{-1} : g_p \cdot \mathbb{V}(\mathcal{A}_x[p^\infty]) \xrightarrow{\sim} \mathbb{V}(\mathcal{A}_x[p^\infty])$, then the corresponding abelian variety $A_{g_p x}$ with induced polarization and level structure, and the same tensors $s_{\alpha,0,x}$ (as usual note g_p preserves tensors).

Since g_p is (the image of) an element of $J_b(\mathbb{Q}_p)$, which consists of *self*-quasi-isogenies, we see that $\mathscr{G}_{g_p x}$ is isomorphic to $\mathcal{A}_x[p^\infty]$, and the quasi-isogeny $\mathcal{A}_x[p^\infty] \to \mathscr{G}_{g_p x}$ corresponds via j to g_p^{-1}. Thus, taking m large enough that $p^m g_p^{-1}$ is an isogeny, we can identify $\mathscr{G}_{g_p x}$ with $\mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1})$ and $\mathcal{A}_{g_p x}$ with $\mathcal{A}_x/j(\ker p^m g_p^{-1})$, with the induced polarization and away-from-p level structure, the same tensors, and the Igusa level structure $g_p^* j$. This matches the data produced by traversing the diagram the other way, and we see that the diagram commutes.

To show the diagram is a pullback, let $x_1 \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ be any point whose image x_1' in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$ is in the isogeny class of x'. We want to show that x_1 is in the isogeny class of x.

Since x_1' and x' lie in the same isogeny class, they are related by a pair $(g^p, g_0) \in G(\mathbb{A}_f^p) \times X_v(b)$. In particular, the p-divisible groups are related by a quasi-isogeny

$\mathcal{A}_x[p^\infty] \to \mathcal{A}_{x_1}[p^\infty]$ corresponding to the isomorphism

$$\mathbb{V}(\mathcal{A}_{x_1}[p^\infty]) \cong g_0 \cdot \mathbb{D}(\mathcal{A}_x[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{V}(\mathcal{A}_x[p^\infty]).$$

Using the Igusa level structures j at x and j_1 at x_1, we can translate this to a quasi-isogeny $\Sigma \to \Sigma$ given by an element $g_p^{-1} \in J_b(\mathbb{Q}_p)$ (i.e. we define g_p to be the inverse of this quasi-isogeny). We claim that x_1 is related to x by $(g^p, g_p) \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$.

Indeed, g_p maps to $g_0 \in X_v(b)$ because by construction they send $\mathbb{D}(\mathcal{A}_x[p^\infty])$ to the same lattice $g_0 \cdot \mathbb{D}(\mathcal{A}_x[p^\infty]) = g_p \cdot \mathbb{D}(\mathcal{A}_x[p^\infty])$ in $\mathbb{V}(\mathcal{A}_x[p^\infty])$. Thus x_1 and $x \cdot (g^p, g_p)$ have the same image in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$, so it only remains to show they have the same Igusa level structure. This is also essentially by construction: g_p was defined to make the following diagram commute

$$\begin{array}{ccc} \Sigma & \xrightarrow{j} & \mathcal{A}_x[p^\infty] \\ {\scriptstyle g_p^{-1}}\downarrow & & \downarrow \\ \Sigma & \xrightarrow[j_1]{} & \mathcal{A}_{x_1}[p^\infty] \end{array}$$

and the Igusa level structure $g_p^* j$ at $x \cdot (g^p, g_p)$ is defined to make the following diagram commute.

$$\begin{array}{ccc} \Sigma & \xrightarrow{j} & \mathcal{A}_x[p^\infty] \\ {\scriptstyle p^m g_p^{-1}}\downarrow & & \downarrow \\ \Sigma & \xrightarrow[g_p^* j]{} & \mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1}) \end{array}$$

Note $\mathcal{A}_{x_1}[p^\infty] = \mathcal{A}_x[p^\infty]/j(\ker p^m g_p^{-1})$, because x_1 and $x \cdot (g^p, g_p)$ have the same associated abelian variety and therefore p-divisible group. Thus (after adjusting the vertical arrows by p^m in the first diagram to make them isogenies), we see that j_1 and $g_p^* j$ both make the diagram commute—and since there is a unique isomorphism making the diagram commute, we conclude $j_1 = g_p^* j$ as desired. $\square$

Recall that in 2.5.5 we have fixed a class $\mathbf{b} \in B(G)$, which specifies the isogeny class of our fixed p-divisible group Σ, and therefore the Newton stratum over which our Igusa variety Ig_Σ lies.

Lemma 3.2.4. *Each isogeny class in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$ is contained in a single Newton stratum. The isogeny classes in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$ which give rise to a non-empty isogeny class in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ are precisely those contained in the $\mathbf{b}$-stratum $\overline{\mathscr{S}}^{(\mathbf{b})}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$.*

Proof. By the moduli description of isogeny classes on the Shimura variety in Proposition 3.1.8, if two points $x_1, x_2 \in \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ lie in the same isogeny class, there is a quasi-isogeny $\mathcal{A}_{x_1} \to \mathcal{A}_{x_2}$ such that the induced map $\mathbb{D}(\mathcal{A}_{x_2}[p^\infty]) \to \mathbb{D}(\mathcal{A}_{x_1}[p^\infty])$ takes $s_{\alpha,0,x_2}$ to $s_{\alpha,0,x_1}$. Thus it induces an isomorphism between the isocrystals with G-structure at x_1 and x_2, which shows that x_1 and x_2 lie in the same Newton stratum. This verifies the first claim.

Since each isogeny class on the Igusa variety is the preimage of an isogeny class on the Shimura variety (Proposition 3.2.3), the isogeny classes in $\overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ which give rise to a non-empty isogeny class in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ are precisely those which intersect the central leaf C_Σ. Thus to prove the claim, it suffices to show that every isogeny class in $\overline{\mathscr{S}}^{(\mathbf{b})}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ intersects the central leaf.

Let $x_1 \in \overline{\mathscr{S}}^{(\mathbf{b})}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ and $x_2 \in C_\Sigma(\overline{\mathbb{F}}_p)$. The point x_2 is auxiliary; we use it to show x_1 is isogenous to a point in C_Σ, but this point may not be x_2.

Choosing $\tilde{x}_1, \tilde{x}_2 \in \mathrm{Sh}_{K_p}(G,X)(L)$ specializing to x_1, x_2, we have isomorphisms

$$\begin{array}{ccccc} \mathbb{D}(\mathcal{A}_{x_1}[p^\infty]) & \longleftrightarrow & H^1_{\mathrm{crys}}(\mathcal{A}_{x_1}/\mathcal{O}_L) & \overset{*}{\longleftrightarrow} & H^1_{\text{ét}}(\mathcal{A}_{\tilde{x}_1,\overline{L}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_L \\ & & & & \updownarrow \\ & & & & V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L \\ & & & & \updownarrow \\ \mathbb{D}(\mathcal{A}_{x_2}[p^\infty]) & \longleftrightarrow & H^1_{\mathrm{crys}}(\mathcal{A}_{x_2}/\mathcal{O}_L) & \overset{*}{\longleftrightarrow} & H^1_{\text{ét}}(\mathcal{A}_{\tilde{x}_2,\overline{L}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_L \end{array}$$

where the arrows labeled "$*$" are given by [Kis17, 1.3.7(2)]. We use these isomorphisms to identify $\mathbb{D}(\mathcal{A}_{x_1}[p^\infty])$ and $\mathbb{D}(\mathcal{A}_{x_2}[p^\infty])$ with $V^*_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$, so that we can name isogenies and Frobenius by elements of $G(L)$.

Now, since x_1 and x_2 lie in the same Newton stratum, we can choose an isogeny $\mathcal{A}_{x_1}[p^\infty] \to \mathcal{A}_{x_2}[p^\infty]$. Using the above identifications, this isogeny is given by an element $g^{-1} \in G(L)$ (i.e. we define g to be the inverse of this isogeny), and

$$\mathbb{D}(\mathcal{A}_{x_2}[p^\infty]) = g \cdot \mathbb{D}(\mathcal{A}_{x_1}[p^\infty]) \subset \mathbb{D}(\mathcal{A}_{x_1}[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

This relates the Frobenius $b_1\sigma$ on $\mathbb{D}(\mathcal{A}_{x_1}[p^\infty])$ and $b_2\sigma$ on $\mathbb{D}(\mathcal{A}_{x_2}[p^\infty])$, regarding $b_1, b_2 \in G(L)$, by

$$g^{-1} b_2 \sigma(g) = b_1.$$

By [Kis17, 1.1.12], we have $b_1 \in G(\mathcal{O}_L) v_0(p) G(\mathcal{O}_L)$, where $v_0 = \sigma(\mu_0^{-1})$ and μ_0^{-1} is a $G_{\mathcal{O}_L}$-valued cocharacter giving the filtration on $\mathbb{D}(\mathcal{A}_{x_1}[p^\infty]) \otimes_{\mathcal{O}_L} \overline{\mathbb{F}}_p$. By [Kis17, 1.3.7(3)], μ_0 is conjugate to μ_h for $h \in X$, and therefore conjugate to the cocharacter μ specified in 3.1.1. This implies that v_0 is conjugate to the cocharacter v of 3.1.1, and since both are $G_{\mathcal{O}_L}$-valued, they are conjugate by an element of $G(\mathcal{O}_L)$.

Thus $G(\mathcal{O}_L)v_0(p)G(\mathcal{O}_L) = G(\mathcal{O}_L)v(p)G(\mathcal{O}_L)$. Furthermore we can choose our basepoint x of 3.1.1 to be x_2, so that the element $b \in G(L)$ we produce is b_2.

Combining these facts, we get

$$g^{-1}b\sigma(g) = b_1 \in G(\mathcal{O}_L)v_0(p)G(\mathcal{O}_L) = G(\mathcal{O}_L)v(p)G(\mathcal{O}_L),$$

which shows that g defines an element of $X_v(b)$. Furthermore, the image of g under the isogeny class map associated to x_1 is a point with Dieudonné module $g \cdot \mathbb{D}(\mathcal{A}_{x_1}[p^\infty]) \cong \mathbb{D}(\mathcal{A}_{x_2}[p^\infty])$. This image is a point in the central leaf which is isogenous to x_1, as desired. □

Finally we summarize the results of this section in the following proposition.

Proposition 3.2.5. *There is a canonical bijection between isogeny classes on* $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ *and isogeny classes on* $\overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$ *contained in the* $\mathbf{b}$*-stratum, given by taking preimage under the map* $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \to \overline{\mathscr{S}}_{K_p}(G,X)(\overline{\mathbb{F}}_p)$.

Proof. This follows from Lemma 3.2.3 and Lemma 3.2.4. □

3.3 b-admissible Morphisms of Galois Gerbs

Recall from 2.5.5 that we have fixed a class $\mathbf{b} \in B(G)$ specifying the Newton stratum our Igusa variety lies over. We saw in §3.2 that isogeny classes on our Igusa variety correspond to a subset of isogeny classes on our Shimura variety, namely those lying in the $\mathbf{b}$-stratum. As in the case of Shimura varieties, we want to parametrize isogeny classes by (conjugacy classes of) admissible morphisms of Galois gerbs. The next definition specifies the subset of admissible morphisms that will correspond to the $\mathbf{b}$-stratum, and therefore to isogeny classes on our Igusa variety.

Definition 3.3.1. A morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is $\mathbf{b}$*-admissible* if it is admissible as in Definition 2.6.9, with a refined condition at p: that $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate to an unramified morphism θ with $[b_\theta] = \mathbf{b}$.

This property is preserved under conjugation by $G(\overline{\mathbb{Q}})$, so we have a well-defined notion of $\mathbf{b}$-admissibility for a conjugacy class $[\phi]$ of admissible morphisms.

3.3.2 If $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is $\mathbf{b}$-admissible, we define a morphism $I_\phi(\mathbb{A}_f) \to G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ as follows. Recall from §2.6 the set

$$X^p(\phi) = \{g = (g_\ell) \in G(\overline{\mathbb{A}}_f^p) : \mathrm{Int}(g_\ell) \circ \xi_\ell = \phi(\ell) \circ \zeta_\ell\}$$

is a $G(\mathbb{A}_f^p)$-torsor, and therefore the natural left-multiplication action of $I_\phi(\mathbb{A}_f^p)$ on $X^p(\phi)$ gives a morphism $I_\phi(\mathbb{A}_f^p) \to G(\mathbb{A}_f^p)$ well-defined up to $G(\mathbb{A}_f^p)$-conjugacy. At p, let $\theta : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ be an unramified morphism conjugate to $\phi(p) \circ \zeta_p$. Then we have maps

$$I_\phi \hookrightarrow I_{\phi(p)\circ\zeta_p} \xrightarrow{\sim} I_\theta \xrightarrow{\sim} J_{b_\theta} \xrightarrow{\sim} J_b,$$

the first isomorphism because θ is conjugate to $\phi(p) \circ \zeta_p$, the second from Lemma 2.6.7, and the third because b_θ is σ-conjugate to b (by the **b**-admissible condition). This gives a map $I_\phi(\mathbb{Q}_p) \to J_b(\mathbb{Q}_p)$ well-defined up to $J_b(\mathbb{Q}_p)$-conjugacy.

We have produced a morphism

$$I_\phi(\mathbb{A}_f) \to G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p).$$

Define

$$S^{\mathrm{Ig}}(\phi) = I_\phi(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p), \tag{3.3.3}$$

where the action of $I_\phi(\mathbb{Q})$ is given by the map above. This is the set that we will use to parametrize the isogeny class $\mathscr{I}^{\mathrm{Ig}}$ corresponding to a conjugacy class $[\phi]$—although we will only identify the action of $I_\phi(\mathbb{Q})$ with $I_x(\mathbb{Q})$ up to a twist, as described in §3.5.

3.4 Kottwitz Triples

To make the connection between isogeny classes and admissible morphisms, we import the technique of [Kis17]. Namely, Kisin constructs a bijection between isogeny classes and (conjugacy classes of) admissible morphisms by making use of intermediate constructions of special point data and Kottwitz triples. We will show in §3.6 that this bijection induces a bijection in our setting as well. In this section we review the necessary details and connect them to our setting.

3.4.1 A *Kottwitz triple of level r* is a triple $\mathfrak{k} = (\gamma_0, \gamma, \delta)$ consisting of

- $\gamma_0 \in G(\mathbb{Q})$ a semi-simple element which is elliptic in $G(\mathbb{R})$,
- $\gamma = (\gamma_\ell)_{\ell \neq p} \in G(\mathbb{A}_f^p)$ conjugate to γ_0 in $G(\overline{\mathbb{A}}_f^p)$, and
- $\delta \in G(\mathbb{Q}_{p^r})$ such that γ_0 is conjugate to $\gamma_p = \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ in $G(\overline{\mathbb{Q}}_p)$;

this data is required to satisfy the further condition (to be explained presently) that

(∗) there is an inner twist I of I_0 over $\mathbb{Q}$ with $I \otimes_{\mathbb{Q}} \mathbb{R}$ anisotropic mod center, and $I \otimes_{\mathbb{Q}} \mathbb{Q}_v$ is isomorphic to I_v as inner twists of I_0 for all finite places v of $\mathbb{Q}$.

Here I_0 is the centralizer of γ_0^n in G, and I_ℓ for $\ell \neq p$ is the centralizer of γ_ℓ^n in $G_{\mathbb{Q}_\ell}$, and I_p is a $\mathbb{Q}_p$-group defined on points by

$$I_p = \{g \in G(W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} R) : g^{-1}\delta\sigma(g) = \delta\};$$

all of these groups stabilize for n sufficiently large, and we mean to take the stabilized group.

A *Kottwitz triple* is an equivalence class of Kottwitz triples of various level, where we take the smallest equivalence relation so that

- two triples $(\gamma_0, \gamma, \delta)$ and $(\gamma_0', \gamma', \delta')$ of the same level r are equivalent if γ_0, γ_0' are conjugate in $G(\overline{\mathbb{Q}})$ and γ, γ' are conjugate in $G(\mathbb{A}_f^p)$ and δ, δ' are σ-conjugate in $\mathbb{Q}_{p^r}$, and
- a triple $(\gamma_0, \gamma, \delta)$ of level r is equivalent to the triple $(\gamma_0^m, \gamma^m, \delta)$ of level rm.

Define a Kottwitz triple $(\gamma_0, \gamma, \delta)$ to be $\mathbf{b}$*-admissible* if the σ-conjugacy class of δ is $\mathbf{b}$. This is clearly seen to be preserved under the equivalences above.

A *refined Kottwitz triple* is a tuple $\tilde{\mathfrak{k}} = (\gamma_0, \gamma, \delta, I, \iota)$ where $\mathfrak{k} = (\gamma_0, \gamma, \delta)$ is a Kottwitz triple (we say $\tilde{\mathfrak{k}}$ is a refinement of $\mathfrak{k}$), and I is a group as in condition $(*)$ above, and $\iota : I \otimes_{\mathbb{Q}} \mathbb{A}_f \to I_{\mathbb{A}_f^p} \times I_p$ is an isomorphism of inner twists of I_0, where $I_{\mathbb{A}_f^p}$ is the centralizer of γ^n in $G_{\mathbb{A}_f^p}$ for n sufficiently large.

We consider two refined Kottwitz triples $(\gamma_0, \gamma, \delta, I, \iota)$ and $(\gamma_0', \gamma', \delta', I', \iota')$ to be equivalent if

- $(\gamma_0, \gamma, \delta)$ is equivalent to $(\gamma_0', \gamma', \delta')$ as Kottwitz triples, so that we can (and do) identify the groups $I_{\mathbb{A}_f^p} \times I_p$ in each case; and
- there is an isomorphism $I \to I'$ over $\mathbb{Q}$ as inner twists of I_0 which intertwines ι, ι'.

The condition $(*)$ determines the inner twist I uniquely up to conjugation by $I(\mathbb{Q})$, so the last condition above is equivalent to requiring that ι, ι' are intertwined up to conjugation by $I(\mathbb{Q})$.

If $\tilde{\mathfrak{k}}$ is a refined Kottwitz triple, then the isomorphism $\iota : I \otimes_{\mathbb{Q}} \mathbb{A}_f \to I_{\mathbb{A}_f^p} \times I_p$ gives an injection

$$I(\mathbb{A}_f) \hookrightarrow G(\mathbb{A}_f^p) \times J_\delta(\mathbb{Q}_p),$$

as $I_{\mathbb{A}_f^p}$ is a subgroup of $G_{\mathbb{A}_f^p}$ and I_p is a subgroup of J_δ. For $\tilde{\mathfrak{k}}$ a $\mathbf{b}$-admissible Kottwitz triple, we define

$$S^{\mathrm{Ig}}(\tilde{\mathfrak{k}}) = I(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p), \tag{3.4.2}$$

where we have used $\mathbf{b}$-admissibility to replace J_δ with the isomorphic group J_b. This is the intermediate set that allows us to connect an isogeny class $\mathscr{I}^{\mathrm{Ig}}$ with our parametrizing set $S^{\mathrm{Ig}}(\phi)$.

3.4.3 Let $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$. We recall how to associate a refined Kottwitz triple to the isogeny class $\mathscr{I}_x^{\mathrm{Ig}}$, following [Kis17, 4.4.6]. The same construction associates a refined Kottwitz triple to an isogeny class $\mathscr{I}^{\mathrm{Sh}}$ on the Shimura variety, and since it is the same construction the triples match for isogeny classes $\mathscr{I}^{\mathrm{Ig}}$ and $\mathscr{I}^{\mathrm{Sh}}$ matched by Proposition 3.2.5.

The level structure η^p at x identifies $G_{\mathbb{Q}_\ell}$ with the subgroup of $\mathrm{GL}(H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell))$ fixing the tensors $\{s_{\alpha,\ell,x}\} \subset H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell)^{\otimes}$, and this allows us to write the geometric Frobenius on $H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell)$ as an element $\gamma_\ell \in G(\mathbb{Q}_\ell)$. Let $\gamma = (\gamma_\ell) \in G(\mathbb{A}_f^p)$.

At p, we similarly have an isomorphism

$$V^\vee_{\mathbb{Z}_{p^r}} \xrightarrow{\sim} \mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^r})$$

which identifies $G_{\mathbb{Q}_p}$ with the subgroup of $\mathrm{GL}(\mathbb{D}(\mathcal{A}_x[p^\infty])(\mathbb{Z}_{p^r}))$ fixing the tensors $\{s_{\alpha,0,x}\}$, and allows us to write the Frobenius on $\mathbb{D}(\mathcal{A}_x[p^\infty])$ as $\delta\sigma$ for some $\delta \in G(\mathbb{Q}_{p^r})$.

With this choice of γ and δ, [Kis17, Cor 2.3.1] states that there is an element $\gamma_0 \in G(\mathbb{Q})$ that makes $(\gamma_0, \gamma, \delta)$ a Kottwitz triple, which we denote $\mathfrak{k}(x)$.

Furthermore if we take $I = I_x \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$ then the identifications above give an isomorphism $\iota : I_x \otimes_{\mathbb{Q}} \mathbb{A}_f \to I_{\mathbb{A}_f^p} \times I_p$ by taking ℓ-adic and crystalline cohomology

$$\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x) \to \mathrm{GL}(H^1_{\text{ét}}(\mathcal{A}_x, \mathbb{Q}_\ell)) \times \mathrm{GL}(H^1_{\text{crys}}(\mathcal{A}_x/\mathbb{Z}_{p^r})).$$

This completes the refined Kottwitz triple

$$\tilde{\mathfrak{k}}(x) = (\gamma_0, \gamma, \delta, I, \iota)$$

associated to the isogeny class $\mathscr{I}_x^{\mathrm{Ig}}$.

Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be an admissible morphism. We refer to [Kis17, 4.5.1] for the construction of a Kottwitz triple $\mathfrak{k}(\phi)$ associated to the conjugacy class $[\phi]$, except to note that the element δ appearing in this Kottwitz triple is a σ-conjugate of the element b_θ produced by an unramified morphism θ conjugate to $\phi(p) \circ \zeta_p$ as in Definition 2.6.9. This Kottwitz triple has a natural refinement $\tilde{\mathfrak{k}}(\phi)$ taking $I = I_\phi$.

We now establish a number of simple compatibilities between isogeny classes, $\mathbf{b}$-admissible morphisms, and their associated Kottwitz triples.

Lemma 3.4.4. *Let $\mathscr{I}^{\mathrm{Sh}}$ be an isogeny class in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$. The associated Kottwitz triple $\mathfrak{k}(\mathscr{I}^{\mathrm{Sh}})$ is $\mathbf{b}$-admissible if and only if $\mathscr{I}^{\mathrm{Sh}}$ is contained in the $\mathbf{b}$-stratum.*

Proof. This essentially follows from the definitions. The Kottwitz triple $\mathfrak{k}(\mathscr{I}^{\mathrm{Sh}}) = (\gamma_0, \gamma, \delta)$ is $\mathbf{b}$-admissible if the σ-conjugacy class of δ is $\mathbf{b}$. Also the construction of this element δ is from the Frobenius on the Dieudonné module of the p-divisible group at a point in $\mathscr{I}^{\mathrm{Sh}}$, and so the σ-conjugacy class of this element records the Newton stratum in which the isogeny class lies. □

Lemma 3.4.5. *For any $x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$, there is a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$\mathscr{I}_x^{\mathrm{Ig}} \cong S^{\mathrm{Ig}}(\tilde{\mathfrak{k}}(x)).$$

Proof. This essentially follows from the definitions. Compare Definition 3.2.2 with (3.4.2), noting that the refined Kottwitz triple $\tilde{\mathfrak{k}}(x)$ is **b**-admissible and has $I = I_x$. □

Lemma 3.4.6. *An admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is **b**-admissible if and only if the associated Kottwitz triple $\mathfrak{k}(\phi) = (\gamma_0, \gamma, \delta)$ is **b**-admissible.*

Proof. This follows essentially from the definitions. Recall that an admissible morphism ϕ is **b**-admissible if the σ-conjugacy class $[b_\theta]$ is the class **b**, where b_θ is the element produced by any unramified morphism θ conjugate to $\phi(p) \circ \zeta_p$. A Kottwitz triple $(\gamma_0, \gamma, \delta)$ is **b**-admissible if the σ-conjugacy class of δ is **b**. But the element δ arising in the Kottwitz triple $\mathfrak{k}(\phi)$ associated to ϕ is a σ-conjugate of b_θ, so these conditions are equivalent. □

Lemma 3.4.7. *For any **b**-admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ and $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, there is a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$S^{\mathrm{Ig}}(\phi) \cong S^{\mathrm{Ig}}(\tilde{\mathfrak{k}}(\phi)).$$

Proof. This essentially follows from the definitions, comparing (3.5.3) with (3.4.2), and noting that $\tilde{\mathfrak{k}}(\phi, \tau)$ has $I = I_\phi$. □

3.5 τ-twists

Our goal is to parametrize each isogeny class $\mathscr{I}^{\mathrm{Ig}} \subset \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ by a set $S^{\mathrm{Ig}}(\phi) = I_\phi(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. However, we will only identify the action of $I_x(\mathbb{Q})$ with $I_\phi(\mathbb{Q})$ up to twisting by an element $\tau \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$. This is still enough for the purposes of point-counting, but requires some care in working with the "τ-twists". We develop the necessary theory in this section, following [KSZ21, §3].

3.5.1 Let ϕ be an admissible morphism, and define

$$\mathcal{H}(\phi) = I_\phi(\mathbb{A}_f) \backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f) / I_\phi^{\mathrm{ad}}(\mathbb{Q}),$$
$$\mathfrak{E}^p(\phi) = I_\phi(\mathbb{A}_f^p) \backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p).$$

These sets can be given the structure of abelian groups (by comparison with certain abelianized cohomology groups, see [KSZ21, 3.2.1, Lemma 3.2.4]). By weak approximation the natural inclusion $I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p) \to I_\phi^{\mathrm{ad}}(\mathbb{A}_f)$ induces a surjection $\mathfrak{E}^p(\phi) \to \mathcal{H}(\phi)$, which allows us to lift an element of $\mathcal{H}(\phi)$ to $I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, or further $I_\phi(\overline{\mathbb{A}}_f^p)$. We will often do this implicitly, writing $\tau \in \Gamma(\mathcal{H})$ and $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, when the ambiguity in the lift is harmless.

Let $\mathcal{AM}$ be the set of admissible morphisms. Since we want to consider assignments of an element of $\mathcal{H}(\phi)$ for all ϕ simultaneously, it is convenient to consider $\mathcal{H}(\phi)$ as the stalks of a sheaf $\mathcal{H}$ on $\mathcal{AM}$ (regarded as a discrete topological space), and similarly $\mathfrak{E}^p(\phi)$ the stalks of a sheaf $\mathfrak{E}^p$. Let $\Gamma(\mathcal{H})$ and $\Gamma(\mathfrak{E}^p)$ be the global sections of these sheaves, so an element $\tau \in \Gamma(\mathcal{H})$ assigns to each admissible morphism ϕ an element $\tau(\phi) \in \mathcal{H}(\phi)$, and similarly for $\mathfrak{E}^p$. The surjections $\mathfrak{E}^p(\phi) \to \mathcal{H}(\phi)$ produce surjections $\mathfrak{E}^p \to \mathcal{H}$ and $\Gamma(\mathfrak{E}^p) \to \Gamma(\mathcal{H})$.

Define an equivalence relation $\phi_1 \approx \phi_2$ if ϕ_1^Δ is conjugate to ϕ_2^Δ by $G(\overline{\mathbb{Q}})$. If $\phi_1 \approx \phi_2$, then there are canonical isomorphisms

$$\mathrm{Comp}_{\phi_1,\phi_2} : \mathcal{H}(\phi_1) \to \mathcal{H}(\phi_2)$$
$$\mathrm{Comp}^{\mathfrak{E}^p}_{\phi_1,\phi_2} : \mathfrak{E}^p(\phi_1) \to \mathfrak{E}^p(\phi_2)$$

satisfying the relations $\mathrm{Comp}_{\phi_2,\phi_3} \circ \mathrm{Comp}_{\phi_1,\phi_2} = \mathrm{Comp}_{\phi_1,\phi_3}$ and $\mathrm{Comp}_{\phi_1,\phi_1} = \mathrm{id}_{\mathcal{H}(\phi_1)}$, and similarly for $\mathfrak{E}^p$. These isomorphisms show that $\mathcal{H}$ and $\mathfrak{E}^p$ are pulled back from sheaves $\mathcal{H}/\approx$ and $\mathfrak{E}^p/\approx$ on $\mathcal{AM}/\approx$, under the natural quotients

$$\mathcal{AM} \to \mathcal{AM}/\mathrm{conj} \to \mathcal{AM}/\approx.$$

Write $\mathcal{H}/\mathrm{conj}$ and $\mathfrak{E}^p/\mathrm{conj}$ for the intermediate pullbacks to $\mathcal{AM}/\mathrm{conj}$, the set of admissible morphisms up to conjugacy.

3.5.2 Let $\Gamma(\mathcal{H})_0$ be the set of global sections of $\mathcal{H}$ that descend to $\mathcal{AM}/\approx$, and $\Gamma(\mathcal{H})_1$ those that descend to $\mathcal{AM}/\mathrm{conj}$, so we have $\Gamma(\mathcal{H})_0 \subset \Gamma(\mathcal{H})_1 \subset \Gamma(\mathcal{H})$. Define $\Gamma(\mathfrak{E}^p)_0 \subset \Gamma(\mathfrak{E}^p)_1 \subset \Gamma(\mathfrak{E}^p)$ similarly. The surjection $\Gamma(\mathfrak{E}^p) \to \Gamma(\mathcal{H})$ induces a surjection $\Gamma(\mathfrak{E}^p)_0 \to \Gamma(\mathcal{H})_0$.

There is one further technical definition we will need, namely the notion of *tori-rationality* of an element of $\Gamma(\mathcal{H})$ or $\Gamma(\mathfrak{E}^p)$. For this we refer to [KSZ21, §3.3]. We will also need the fact [KSZ21, Lemma 3.3.3] that an element of $\Gamma(\mathcal{H})$ is tori-rational exactly when one (equivalently, every) lift to $\Gamma(\mathfrak{E}^p)$ is tori-rational.

Let $\tau \in \Gamma(\mathcal{H})_1$. Define

$$S_\tau^{\mathrm{Ig}}(\phi) = I_\phi(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p), \tag{3.5.3}$$

as in (3.3.3), except the action of $I_\phi(\mathbb{Q})$ composed with the action of $\tau(\phi)$. Likewise define $\tilde{\mathfrak{k}}(\phi,\tau)$ to be the refined Kottwitz triple obtained from $\tilde{\mathfrak{k}}(\phi)$ replacing ι :

$I_\phi \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} I_{\mathbb{A}_f^p} \times I_p$ by its composition with the action of τ. By taking $\tau \in \Gamma(\mathcal{H})_1$ we ensure that these definitions only depend on the conjugacy class of ϕ. Then we have an immediate analogue of Lemma 3.4.7.

Lemma 3.5.4. *Let $\tau \in \Gamma(\mathcal{H})$. For any **b**-admissible morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$, there is a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$S_\tau^{\mathrm{Ig}}(\phi) \cong S^{\mathrm{Ig}}(\tilde{\mathfrak{k}}(\phi, \tau)).$$

3.6 Langlands-Rapoport-τ Conjecture for Igusa Varieties of Hodge Type

In this section we complete the proof of the first main Theorem 3.6.2, our analogue of the Langlands-Rapoport conjecture for Igusa varieties. This amounts to combining the ingredients developed in the preceding portions of §3.

Proposition 3.6.1. *There exists a bijection*

$$\left\{\begin{matrix}\textit{isogeny classes}\\ \textit{in } \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{conj. classes of}\\ \mathbf{b}\textit{-admissible}\\ \textit{morphisms } \phi\end{matrix}\right\},$$

compatible with (unrefined) Kottwitz triples. Furthermore, there is a tori-rational element $\tau \in \Gamma(\mathcal{H})_0$ making such a bijection compatible with refined Kottwitz triples, in the sense that if $\mathscr{I}_x$ maps to $[\phi]$, then $\tilde{\mathfrak{k}}(x)$ is equivalent to $\tilde{\mathfrak{k}}(\phi, \tau(\phi))$.

Proof. In [Kis17], Kisin constructs a bijection

$$\left\{\begin{matrix}\text{isogeny classes in}\\ \overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conj. classes}\\ \text{of admissible}\\ \text{morphisms } \phi\end{matrix}\right\},$$

compatible with Kottwitz triples (that is, corresponding elements give rise to equivalent Kottwitz triples). This bijection induces a bijection in our case, as we now explain.

By Proposition 3.2.5 we can consider the set of isogeny classes in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ as a subset of the set of isogeny classes in $\overline{\mathscr{S}}_{K_p}(G, X)(\overline{\mathbb{F}}_p)$, namely the subset of isogeny classes contained in the **b**-stratum. By Lemma 3.4.4, this subset is characterized as those isogeny classes whose associated Kottwitz triple is **b**-admissible.

On the other side, the **b**-admissible condition on admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_G$ is a strengthening of the admissible condition, so we can regard the set of conjugacy classes of **b**-admissible morphisms as a subset of the set of conjugacy classes of admissible morphisms. By Lemma 3.4.6, this subset is characterized as those conjugacy classes whose associated Kottwitz triple is **b**-admissible.

Because the bijection above is compatible with Kottwitz triples, it induces a bijection between the subsets on each side corresponding to **b**-admissible Kottwitz triples, and this gives our desired bijection.

It remains to settle the last claim regarding the τ-twist. The bijection between Shimura isogeny classes and conjugacy classes of admissible morphisms constructed in [Kis17] is not known to preserve refined Kottwitz triples, and so we can only identify the refined Kottwitz triples on each side up to a τ-twist. That is, if $\mathscr{I}_x$ maps to $[\phi]$, then $\tilde{\mathfrak{k}}(x)$ is equivalent to $\tilde{\mathfrak{k}}(\phi, \tau(\phi))$ for some $\tau \in \Gamma(\mathcal{H})$.

In [KSZ21] it is conjectured that it should be possible to take τ to be tori-rational and to lie in $\Gamma(\mathcal{H})_0$ (part of their Conjecture 3.4.4), and one of the main results of their paper is to prove this conjecture in the case of abelian type (their Theorem 8.3.3). By the same argument as above, their result carries over to our case, which completes the proof. □

Theorem 3.6.2. *There exists a tori-rational element $\tau \in \Gamma(\mathcal{H})_0$ admitting a $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijection*

$$\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]} S_\tau^{\mathrm{Ig}}(\phi),$$

where the disjoint union ranges over conjugacy classes of **b***-admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$.*

Proof. By Proposition 3.6.1, the conjugacy classes $[\phi]$ parametrizing the disjoint union are in bijection with the set of isogeny classes in $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$. Furthermore there is a tori-rational element $\tau \in \Gamma(\mathcal{H})_0$ such that for corresponding $\mathscr{I}_x$ and $[\phi]$, the refined Kottwitz triples $\tilde{\mathfrak{k}}(x)$ and $\tilde{\mathfrak{k}}(\phi, \tau(\phi))$ are equivalent.

Combining the equivalence of these refined Kottwitz triples with Lemma 3.4.5 and Lemma 3.5.4, we have $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-equivariant bijections

$$\mathscr{I}_x \cong S^{\mathrm{Ig}}(\tilde{\mathfrak{k}}(x)) \cong S^{\mathrm{Ig}}(\tilde{\mathfrak{k}}(\phi, \tau)) \cong S_\tau^{\mathrm{Ig}}(\phi).$$

Thus we have produced bijections between the set of isogeny classes and the set of conjugacy classes of **b**-admissible morphisms, and bijections between each isogeny class and the set $S_\tau^{\mathrm{Ig}}(\phi)$ produced by its corresponding admissible morphism. Together, these produce a bijection $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{[\phi]} S_\tau^{\mathrm{Ig}}(\phi)$ as desired. □

4 Point-Counting Formula for Igusa Varieties of Hodge Type

Our analogue of the Langlands-Rapoport conjecture will now be put to use in finding a trace formula for the cohomology of Igusa varieties of Hodge type, expressing the trace of certain operators on this cohomology in terms of orbital integrals on our algebraic groups. Its precise role will be to give a concrete form for the fixed points of correspondences that appear when computing traces on cohomology via Fujiwara's fixed point theorem. This will result in a trace formula parametrized by Galois gerbs, which must finally be translated into group-theoretic data.

In §4.1 we review Fujiwara's fixed point theorem, packaged in the notion of acceptable functions. In §4.2 we carry out the application of Fujiwara's theorem to arrive at a trace formula parametrized by Galois gerbs, or more precisely, LR pairs. In §4.3 we formalize LR pairs and introduce Kottwitz parameters, which provide the group theoretic language on which the final trace formula will be based. In §4.4 we determine the relationship between Kottwitz parameters and LR pairs that will allow us to translate the formula. In §4.5 we carry out this translation and arrive at our Main Theorem 4.5.17, the final unstable trace formula for Igusa varieties of Hodge type.

We continue to use the notation of the previous sections. In particular, we have fixed a Shimura datum (G, X) of Hodge type, a class $\mathbf{b} \in B(G, \mu^{-1})$, and representative $b \in G(L)$ of this class, which data defines an Igusa variety Ig_Σ.

4.1 Acceptable Functions and Fujiwara's Trace Formula

In order to use Fujiwara's theorem in our context, we introduce certain operators on cohomology and associated correspondences on our Igusa variety.

4.1.1 The (right) action of $G(\mathbb{A}_f^p) \times S_b$ on Ig_Σ extends to an (left) action of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ on $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ as in (2.5.8). These representations are in general infinite-dimensional, so instead of considering traces of group elements, we define operators as follows. Let $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, a smooth (i.e. locally constant) compactly supported function on $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. We define an operator on $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$, also called f by abuse of notation, by

$$v \mapsto \int_{G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)} f(x) x \cdot v \, dx.$$

This operator has finite rank (as we see just below), so it makes sense to ask for its trace. Write

$$\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_i (-1)^i \operatorname{tr}(f \mid H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi));$$

our eventual goal is to find a formula for this trace in the case of a certain class of test functions, namely acceptable functions, which we define in this section.

Any $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ is a finite linear combination of indicator functions $\mathbb{1}_{UgU}$ for $U \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ compact open and $g \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$, so by linearity of trace it suffices to consider $f = \mathbb{1}_{UgU}$. Writing $UgU = \coprod_i g_i g U$ for some finite collection $g_i \in U$, the action of $\mathbb{1}_{UgU}$ on $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ is given by

$$\begin{aligned} \int_{G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)} \mathbb{1}_{UgU}(x) x \cdot v \, dx &= \int_{UgU} x \cdot v \, dx \\ &= \sum_i \int_{g_i g U} x \cdot v \, dx \\ &= \sum_i g_i g \int_U x \cdot v \, dx. \end{aligned}$$

Now $\int_U x \cdot v \, dx$ is $\operatorname{vol}(U)$ times projection to $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)^U$, which is finite-dimensional because $H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)$ is an admissible $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$-representation. This verifies that these operators have finite rank, and realizes the action of $\mathbb{1}_{UgU}$ as $\operatorname{vol}(U)$ times the operator

$$v \mapsto \sum_i g_i g \cdot v \qquad \text{on} \qquad H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)^U.$$

We call this operator $v \mapsto \sum_i g_i g \cdot v$ the *double coset operator* $[UgU]$.

Define

$$U_p(m) = \ker(\operatorname{Aut}(\Sigma, \lambda_\Sigma, \{s_{\alpha,\Sigma}\}) \to \operatorname{Aut}(\Sigma[p^m], \lambda_\sigma, \{s_{\alpha,\Sigma}\})) \subset J_b(\mathbb{Q}_p).$$

These subsets form a neighborhood basis of the identity in $J_b(\mathbb{Q}_p)$, so we can assume $U = U^p \times U_p(m)$ for $U^p \subset G(\mathbb{A}_f^p)$ compact open. Then

$$H_c^i(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)^U = H_c^i(\mathrm{Ig}_{\Sigma, U^p, m}, \mathscr{L}_\xi).$$

In particular, the action of $\mathbb{1}_{UgU}$ is $\operatorname{vol}(U)$ times the double coset action $[UgU]$, and we have (taking the alternating sum over i)

$$\operatorname{tr}(\mathbb{1}_{UgU} \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \operatorname{vol}(U) \operatorname{tr}([UgU] \mid H_c(\mathrm{Ig}_{\Sigma, U^p, m}, \mathscr{L}_\xi)). \tag{4.1.2}$$

4.1.3 Now assume $g = g^p \times g_p \in G(\mathbb{A}_f^p) \times S_b$ (recall S_b from 2.2.7), so that we can consider the action of g on finite-level Igusa varieties. Then the double coset action $[UgU]$ is induced by a correspondence on $\mathrm{Ig}_{\Sigma,U^p,m}$, which allows us to compute the trace by Fujiwara's formula.

We start by considering the set-theoretic correspondence

$$\begin{array}{ccccc} & & \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/(U \cap gUg^{-1}) & & \\ & \swarrow^{[\cdot 1]} & & \searrow^{[\cdot g]} & \\ \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U & & & & \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U \end{array}$$

where the left arrow is the projection to $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U$ and the right arrow is the projection to $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/gUg^{-1}$ followed by (right) multiplication by g. The multi-valued function given by composing the "inverse" of the left arrow with the right arrow is $\sum_i g_i g$ where $U = \coprod_i g_i(U \cap gUg^{-1})$. But the same set of g_i give the coset decomposition $UgU = \coprod_i g_i gU$, so this is the same as the double coset action $[UgU]$.

However, the set map $[\cdot g]$ in the above correspondence may not arise from a map of varieties. In order to get an action on cohomology and apply Fujiwara's theorem to compute the trace, we need to upgrade our set-theoretic correspondence to an algebro-geometric correspondence.

Choose U_1^p and m_1 so that $U_1 = U_1^p \times U_p(m_1) \subset U \cap gUg^{-1}$ and $m_1 - e_1(g_p) \geq m$ (where $e_1(g_p)$ is defined as in 2.2.7); under these conditions g does indeed give a morphism of varieties $\mathrm{Ig}_{\Sigma,U_1^p,m_1} \to \mathrm{Ig}_{\Sigma,U^p,m}$. Consider the correspondence

$$\begin{array}{ccccc} & & \mathrm{Ig}_{\Sigma,U_1^p,m_1} & & \\ & \swarrow^{[\cdot 1]} & & \searrow^{[\cdot g]} & \\ \mathrm{Ig}_{\Sigma,U^p,m} & & & & \mathrm{Ig}_{\Sigma,U^p,m} \end{array}$$

where the left arrow is the projection

$$\mathrm{Ig}_{\Sigma,U_1^p,m_1} \to \mathrm{Ig}_{\Sigma,U^p,m}$$

and the right arrow is projection followed by g-action

$$\mathrm{Ig}_{\Sigma,U_1^p,m_1} \to \mathrm{Ig}_{\Sigma,g^pU^p(g^p)^{-1},m_1} \xrightarrow{g} \mathrm{Ig}_{\Sigma,U^p,m}.$$

Since $U_1 \subset U \cap gUg^{-1}$ this algebro-geometric correspondence induces the same action as the above set-theoretic correspondence, up to a constant factor, namely the index $[U \cap gUg^{-1} : U_1]$. That is, the trace of the action of this algebro-geometric correspondence on $H_c(\mathrm{Ig}_{\Sigma,U^p,m}, \mathscr{L}_\xi)$ is

$$\begin{aligned} &\mathrm{tr}(\text{alg.-geo. correspondence} \mid H_c(\mathrm{Ig}_{\Sigma,U^p,m}, \mathscr{L}_\xi)) \\ &= [U \cap gUg^{-1} : U_1]\,\mathrm{tr}([UgU] \mid H_c(Ig_{\Sigma,U^p,m}, \mathscr{L}_\xi)). \end{aligned} \tag{4.1.4}$$

Definition 4.1.5. Define the fixed point set of the above correspondence by

$$\mathrm{Fix}(UgU) = \{x \in \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/(U \cap gUg^{-1}) : x = xg \text{ in } \mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U\}.$$

(Note that we use the set-theoretic correspondence instead of the algebro-geometric version—this introduces another factor of $[U \cap gUg^{-1} : U_1]$ into Fujiwara's formula, which cancels the factor in (4.1.4)).

Fujiwara's formula gives an expression for the trace of a correspondence in terms of the fixed points of the correspondence. We package Fujiwara's formula in our definition of acceptable functions. Recall from §2.2 the notion of an acceptable element of $J_b(\mathbb{Q}_p)$.

Definition 4.1.6. A function $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ is *acceptable* if

1. for all $(g, \delta) \in \operatorname{supp} f$, we have $\delta \in S_b$ and δ is acceptable;
2. there is a sufficiently small compact open subgroup $U = U^p \times U_p(m) \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ and a finite subset $I \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ such that $f = \sum_{g \in I} \mathbb{1}_{UgU}$; and for each term in this sum, we have
   (a) $\mathrm{Fix}(UgU)$ is finite, and
   (b) the trace of the correspondence on cohomology is given by Fujiwara's formula:

$$\mathrm{tr}([UgU] \mid H_c(\mathrm{Ig}_{\Sigma,U^p,m}, \mathscr{L}_\xi)) = \sum_{x \in \mathrm{Fix}(UgU)} \mathrm{tr}([UgU] \mid (\mathscr{L}_\xi)_x). \tag{4.1.7}$$

The next two lemmas verify that working with acceptable functions is enough for our purposes. Lemma 4.1.8 says that any locally constant compactly supported function can be twisted to be acceptable, and Lemma 4.1.9 says that traces of acceptable functions suffice to determine a representation of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ up to semi-simplification. These are slightly adjusted versions of Lemmas 6.3 and 6.4 of [Shi09].

Lemma 4.1.8. *For any $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, the function $f^{(m,n)}$ defined by $f^{(m,n)}(x) := f(x \cdot p^m (fr^s)^n)$ is an acceptable function for sufficiently large m, n.*

Proof. To show condition 1 of Definition 4.1.6, note that $\operatorname{supp} f^{(m,n)} = p^{-m}(fr^s)^{-n} \operatorname{supp} f$. Lemma 2.2.4 shows that taking n large enough ensures f is supported on elements (g, δ) with δ acceptable, and a similar argument shows that taking m, n large enough ensures that $\delta \in S_b$: increase m to make δ^{-1} an isogeny, and increase n to make $f_{i-1}(\delta) \geq e_i(\delta)$ for all i—compact support allows us to make a uniform choice of m, n that works for the whole support.

For condition 2, as in [Shi09] we choose a model $\mathcal{J}_{\Sigma,U^p,m}$ of $\mathrm{Ig}_{\Sigma,U^p,m}$ over $\mathbb{F}_{q^s}$ such that the isomorphism $\mathcal{J}_{\Sigma,U^p,m} \times_{\mathbb{F}_{q^s}} \overline{\mathbb{F}}_p \xrightarrow{\sim} \mathrm{Ig}_{\Sigma,U^p,m}$ identifies the actions of $F^s_{\mathrm{ab}} \times 1$ on the left hand side and fr^{-s} on the right hand side (here F^s_{ab} is the absolute q^s-power Frobenius on $\mathcal{J}_{\Sigma,U^p,m}$). This is possible because our fixed p-divisible group Σ is completely slope divisible, which implies as in 2.3.2 that Σ is defined over a finite field. Then Fujiwara's trace formula [Fuj97, Var07] implies that condition 2 can be ensured by twisting the correspondence by a high enough power of $F^s_{\mathrm{ab}} \times 1$. This corresponds to twisting our function f by a high enough power n—note that $\mathbb{1}_{UgU}{}^{(m,n)} = \mathbb{1}_{Up^{-m}(fr^s)^{-n}gU}$, so the positive power $(fr^s)^n$ in the definition of $f^{(m,n)}$ results in twisting the correspondence by a negative power $(fr^s)^{-n}$. Thus, increasing n if necessary, we ensure that $f^{(m,n)}$ is an acceptable function. □

Write $\mathrm{Groth}(G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p))$ for the Grothendieck group of admissible representations of $G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p)$.

Lemma 4.1.9. *If $\Pi_1, \Pi_2 \in \mathrm{Groth}(G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p))$ satisfy $\mathrm{tr}(f \mid \Pi_1) = \mathrm{tr}(f \mid \Pi_2)$ for all acceptable functions f, then $\Pi_1 = \Pi_2$ as elements of the Grothendieck group.*

Proof. It suffices to show that if $\mathrm{tr}(f \mid \Pi_1) = \mathrm{tr}(f \mid \Pi_2)$ for all acceptable functions f, then $\mathrm{tr}(f \mid \Pi_1) = \mathrm{tr}(f \mid \Pi_2)$ for all $f \in C^\infty_c(G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p))$. This follows from examining the effects of twisting $f^{(m,n)}$ as in the previous Lemma 4.1.8.

Suppose all acceptable functions have the same trace on Π_1 and Π_2, and let $f \in C^\infty_c(G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p))$. As in 4.1.1, f is a linear combination of indicator functions $\mathbb{1}_{UgU}$, the trace of which can be computed on the subspace of U-invariants. Since Π_1, Π_2 are admissible, their U-invariant subspaces are finite-dimensional for any $U \subset G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p)$ compact open, and therefore there are only finitely many irreducible constituents of Π_1, Π_2 on which the trace of f is non-zero. Denote these constituents by $\{\pi_i\}$, and their multiplicity in Π_1 and Π_2 by a_i and b_i respectively, so that

$$\mathrm{tr}(f \mid \Pi_1) = \sum_i a_i \,\mathrm{tr}(f \mid \pi_i), \qquad \mathrm{tr}(f \mid \Pi_2) = \sum_i b_i \,\mathrm{tr}(f \mid \pi_i).$$

Since π_i is irreducible, the center of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ acts by scalars. Let α_i and β_i be the scalars recording the action of the central elements p and fr^s, respectively. We compute the action of $f^{(m,n)}$ on π_i as follows (integrals are over $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$):

$$\begin{aligned} f^{(m,n)} \cdot v &= \int f^{(m,n)}(x)\; x \cdot v\; dx \\ &= \int f(x)\; (xp^{-m}(fr^s)^{-n}) \cdot v\; dx \\ &= \alpha_i^{-m}\beta_i^{-n} \int f(x)\; x \cdot v\; dx \\ &= \alpha_i^{-m}\beta_i^{-n} f \cdot v. \end{aligned}$$

In particular,

$$\operatorname{tr}(f^{(m,n)} \mid \pi_i) = \alpha_i^{-m}\beta_i^{-n} \operatorname{tr}(f \mid \pi_i).$$

By Lemma 4.1.8, the function $f^{(m,n)}$ is acceptable for all sufficiently large m, n. Thus we have

$$\sum_i \alpha_i^{-m}\beta_i^{-n} a_i \operatorname{tr}(f \mid \pi_i) = \operatorname{tr}(f^{(m,n)} \mid \Pi_1) = \operatorname{tr}(f^{(m,n)} \mid \Pi_2) = \sum_i \alpha_i^{-m}\beta_i^{-n} b_i \operatorname{tr}(f \mid \pi_i)$$

for all sufficiently large m, n. An elementary argument shows that if $\sum_i \alpha_i^{-m}\beta_i^{-n} x_i = \sum_i \alpha_i^{-m}\beta_i^{-n} y_i$ holds for all sufficiently large m, n, it must hold for *all* m, n, and we get

$$\operatorname{tr}(f \mid \Pi_1) = \sum_i a_i \operatorname{tr}(f \mid \pi_i) = \sum_i b_i \operatorname{tr}(f \mid \pi_i) = \operatorname{tr}(f \mid \Pi_2)$$

as desired. □

4.2 Preliminary Point-Counting

In this section we convert our description of $\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ in 3.6.2 to a preliminary form of the trace formula. The main thrust is applying Milne's combinatorial lemma (Lemma 4.2.2 below) to describe the sets $\mathrm{Fix}(UgU)$ appearing in Fujiwara's formula; but this requires some care, and a number of technical lemmas. The main result of the section is Proposition 4.2.10, containing the preliminary form of the trace formula. In order to ease the exposition, we defer the more technical lemmas to the end of the section.

Let f be an acceptable function; our goal is a preliminary formula for $\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi))$. We assume for the moment that f is of the form $\mathbb{1}_{UgU}$ (for $U = U^p \times U_p(m)$) and satisfies condition 2 of Definition 4.1.6. Combining (4.1.2) and

(4.1.7), we find

$$\mathrm{tr}(\mathbb{1}_{UgU} \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \mathrm{vol}(U) \sum_{x \in \mathrm{Fix}(UgU)} \mathrm{tr}([UgU] \mid (\mathscr{L}_\xi)_x). \tag{4.2.1}$$

We proceed to analyze the set $\mathrm{Fix}(UgU)$ (Definition 4.1.5) parametrizing the sum. Recall Milne's combinatorial lemma.

Lemma 4.2.2 ([Mil92, Lemma 5.3]). *Suppose to be given*

- *X, Y sets with left action from a group I,*
- *$a, b : Y \to X$ maps equivariant for the action of I, and*
- *a subgroup $C \subset Z(I)$ satisfying*
  - *the stabilizer in I of every point of $a(Y)$ is C, and*
  - *$I^{\mathrm{der}} \cap C = \{1\}$.*

Then the set $(I \backslash Y)^{a=b}$ of points on which the maps $a, b : I\backslash Y \to I \backslash X$ agree can be written

$$(I\backslash Y)^{a=b} = \coprod_h I_h \backslash Y_h$$

where

- *h ranges over (representatives of) conjugacy classes in I/C,*
- *I_h is the centralizer of h in I, and*
- *$Y_h = \{y \in Y : hay = by\}$.*

From Theorem 3.6.2 we see that for any compact open $U \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ we have a bijection

$$\mathrm{Ig}_\Sigma(\overline{\mathbb{F}}_p)/U \xrightarrow{\sim} \coprod_{[\phi]} I_\phi(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U, \tag{4.2.3}$$

where $[\phi]$ ranges over conjugacy classes of **b**-admissible morphisms, and with the action of $I_\phi(\mathbb{Q})$ possibly twisted by a tori-rational element $\tau \in \Gamma(\mathcal{H})_0$. Recall from 3.5 that we can lift $\tau(\phi) \in \mathcal{H}(\phi)$ to an element of $I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, which we will also call $\tau(\phi)$ by abuse of notation.

For each $[\phi]$, we apply Lemma 4.2.2 with

- $I = I_\phi(\mathbb{Q})$,

- $Y = G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/(U \cap gUg^{-1})$,
- $X = G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U$,
- $a : Y \to X$ given by $x \mapsto x \bmod U$,
- $b : Y \to X$ given by $x \mapsto xg \bmod U$,
- $C = 1$; or in other words, $C = Z_G(\mathbb{Q}) \cap U$ (the intersection taken in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$), which is trivial for sufficiently small U (cf. Lemma 4.2.12).

That C satisfies the conditions of Lemma 4.2.2 is verified in Lemmas 4.2.11 and 4.2.12.

Then, on the one hand,

$$(I\backslash Y)^{a=b}$$
$$= \{x \in I_\phi(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/(U \cap gUg^{-1}) : x = xg \text{ in } I_\phi(\mathbb{Q})\backslash G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U\}$$

is the $[\phi]$-component of $\mathrm{Fix}(UgU)$ under the bijection (4.2.3); on the other hand, Lemma 4.2.2 gives

$$(I\backslash Y)^{a=b}$$
$$= \coprod_{\varepsilon} I_{\phi,\varepsilon}(\mathbb{Q})\backslash\{x \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/(U \cap gUg^{-1}) : \varepsilon x = xg \text{ in } G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U\}$$

where ε runs over (representatives of) conjugacy classes in $I_\phi(\mathbb{Q})$. Thus we can write

$$\mathrm{Fix}(UgU) =$$
$$\coprod_{[\phi]} \coprod_{\varepsilon} I_{\phi,\varepsilon}(\mathbb{Q})\backslash\{x \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/(U \cap gUg^{-1}) : \varepsilon x = xg \text{ in } G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U\}. \tag{4.2.4}$$

We can count the cardinality of the (ϕ, ε) component as follows. To render the formulas more readable, we abbreviate

$$G_b := G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p).$$

For an element $y = y^p \times y_p \in G_b$ and function $f \in C_c^\infty(G_b)$, we define an orbital integral

$$O_y^{G_b}(f) = \int_{G_{b,y}^\circ \backslash G_b} f(x^{-1}yx)\, dx,$$

where $G_{b,y}^\circ = G_{y^p}^\circ(\mathbb{A}_f^p) \times (J_b)_{y_p}^\circ(\mathbb{Q}_p)$.

Note that the condition "$\varepsilon x = xg$ in G_b/U" appearing in (4.2.4) can be rewritten as "$x^{-1}\varepsilon x \in gU$". Then (ignoring τ-twists for the moment) the number of cosets $x \in G_b/(U \cap gUg^{-1})$ satisfying this property would be counted by the orbital integral

$$O_\varepsilon^{G_b}(\mathbb{1}_{gU}) = \int_{G_{b,\varepsilon}^\circ \backslash G_b} \mathbb{1}_{gU}(x^{-1}\varepsilon x)\, dx;$$

or rather, each coset would contribute $\mathrm{vol}\left((G_{b,\varepsilon}^\circ \cap U \cap gUg^{-1})\backslash(U \cap gUg^{-1})\right)$ to this orbital integral.

To properly count the (ϕ, ε) component we need to account for the possible twist of the action of $I_{\phi,\varepsilon}(\mathbb{Q})$ by $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$. Write $\gamma \times \delta$ for the image of $\mathrm{Int}(\tau(\phi))\varepsilon$ in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ (we will later formalize and interpret this as a *classical Kottwitz parameter*).

We can rewrite the (ϕ, ε) component of $\mathrm{Fix}(UgU)$ in (4.2.4) as

$$I_{\phi,\varepsilon}(\mathbb{Q})\backslash\{x \in G_b/(U \cap gUg^{-1}) : (\gamma \times \delta)x = xg \text{ in } G_b/U\},$$

where the embedding of $I_{\phi,\varepsilon}(\mathbb{Q})$ into G_b is twisted by $\mathrm{Int}(\tau(\phi))$. In particular, under this embedding the image of $I_{\phi,\varepsilon}$ in G_b lies in $G_{b,\gamma\times\delta}$.

Now a detailed computation of the (ϕ, ε) component goes as follows:

$$\begin{aligned}
&\left| I_{\phi,\varepsilon}(\mathbb{Q})\backslash\{x \in G_b/(U \cap gUg^{-1}) \mid \varepsilon x = xg \text{ in } G_b/U\} \right| \qquad (4.2.5)\\
&= \left| I_{\phi,\varepsilon}(\mathbb{Q})\backslash\{x \in G_b/(U \cap gUg^{-1}) : (\gamma \times \delta)x = xg \text{ in } G_b/U\} \right| \\
&= \frac{1}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I_{\phi,\varepsilon}^\circ(\mathbb{Q})]} \left| I_{\phi,\varepsilon}^\circ(\mathbb{Q})\backslash\{x \in G_b/(U \cap gUg^{-1}) \mid (\gamma \times \delta)x = xg \text{ in } G_b/U\} \right| \\
&= \frac{1}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I_{\phi,\varepsilon}^\circ(\mathbb{Q})]\, \mathrm{vol}(U \cap gUg^{-1})} \int_{I_{\phi,\varepsilon}^\circ(\mathbb{Q})\backslash G_b} \mathbb{1}_{gU}(x^{-1}(\gamma \times \delta)x) dx \\
&= \frac{\mathrm{vol}\left(I_{\phi,\varepsilon}^\circ(\mathbb{Q})\backslash G_{b,\gamma\times\delta}^\circ\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I_{\phi,\varepsilon}^\circ(\mathbb{Q})]\, \mathrm{vol}(U \cap gUg^{-1})} \int_{G_{b,\gamma\times\delta}^\circ\backslash G_b} \mathbb{1}_{gU}(x^{-1}(\gamma \times \delta)x) dx \\
&= \frac{\mathrm{vol}\left(I_{\phi,\varepsilon}^\circ(\mathbb{Q})\backslash G_{b,\gamma\times\delta}^\circ\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I_{\phi,\varepsilon}^\circ(\mathbb{Q})]\, \mathrm{vol}(U \cap gUg^{-1})} O_{\gamma\times\delta}^{G_b}(\mathbb{1}_{gU}). \qquad (4.2.6)
\end{aligned}$$

In the second and third equalities we implicitly use the fact that $I_{\phi,\varepsilon}(\mathbb{Q})$ acts freely on $G_b/(U \cap gUg^{-1})$—less obviously in the second, where we use the fact that a coset of $U \cap gUg^{-1}$ in $I_{\phi,\varepsilon}^\circ(\mathbb{Q})\backslash G_b$ has volume $\mathrm{vol}(U \cap gUg^{-1})$, i.e. the same volume as in G_b. That this action is indeed free is verified in Lemma 4.2.13 below.

The value of the expression (4.2.6) depends only on the double coset UgU, rather than the single coset gU; indeed, if we replace g with ug for some $u \in U$, then sending x to xu^{-1} gives a bijection between the corresponding sets in (4.2.5). Write $UgU = \coprod_i g_i gU$. Then we can average (4.2.6) over the set of cosets $g_i gU$ in UgU, and the result will be equal to the original (4.2.6) because all terms in the average are in fact equal. Thus:

$$\begin{aligned}
&\left| I_{\phi,\varepsilon}(\mathbb{Q})\backslash\{x \in G_b/(U \cap gUg^{-1}) \mid \varepsilon x = xg \text{ in } G_b/U\}\right| \\
&= \frac{1}{[UgU : U]} \sum_i \frac{\operatorname{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q})\backslash G^\circ_{b,\gamma\times\delta}\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})] \operatorname{vol}(U \cap g_i gUg^{-1}g_i^{-1})} O^{G_b}_{\gamma\times\delta}(\mathbb{1}_{g_i gU}) \\
&= \frac{\operatorname{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q})\backslash G^\circ_{b,\gamma\times\delta}\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})] \operatorname{vol}(U)} \sum_i O^{G_b}_{\gamma\times\delta}(\mathbb{1}_{g_i gU}) \\
&= \frac{\operatorname{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q})\backslash G^\circ_{b,\gamma\times\delta}\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})] \operatorname{vol}(U)} O^{G_b}_{\gamma\times\delta}(\mathbb{1}_{UgU}). \qquad (4.2.7)
\end{aligned}$$

We have written $[UgU : U]$ for the number of cosets $g_i gU$ in UgU, and in the second equality used the fact that this is equal to the number of cosets of $U \cap gUg^{-1}$ in U, so that $[UgU : U] \operatorname{vol}(U \cap gUg^{-1}) = \operatorname{vol}(U)$. (Of course also $\operatorname{vol}(U \cap g_i gUg^{-1}g_i^{-1}) = \operatorname{vol}(U \cap gUg^{-1})$ as they are conjugate by g_i).

The formula (4.2.7) will serve as our preliminary count of the (ϕ, ε) component of $\operatorname{Fix}(UgU)$. In order to take this expression further, we will need to develop some theory on LR pairs and Kottwitz parameters, which we do in §4.3.

Returning now to our starting point (4.2.1), we compute the local terms arising in Fujiwara's formula.

Lemma 4.2.8. *The naive local term* $\operatorname{tr}([UgU] \mid (\mathscr{L}_\xi)_y)$ *at any point* y *in the* (ϕ, ε) *component of* $\operatorname{Fix}(UgU)$ *is* $\operatorname{tr}(\xi(\varepsilon))$.

Here we are regarding $\varepsilon \in G(\overline{\mathbb{Q}})$ via the natural inclusion $I_\phi(\mathbb{Q}) \subset G(\overline{\mathbb{Q}})$.

Proof. Let $y \in \operatorname{Ig}_\Sigma(\overline{\mathbb{F}}_p)/(U \cap gUg^{-1})$ be a point in $\operatorname{Fix}(UgU)$, and let $\tilde{y} \in \operatorname{Ig}_\Sigma(\overline{\mathbb{F}}_p)$ be a lift of y. Then the fact that y is a fixed point implies $\tilde{y}g = \tilde{y}u$ for some $u \in U$. As in [Kot92, p.433] the local term at y is then $\operatorname{tr}(\xi(gu^{-1}))$.

If y belongs to the (ϕ, ε) component of $\operatorname{Fix}(UgU)$ in the decomposition (4.2.4), then corresponding to y there is a point $x \in G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p)$ satisfying $\varepsilon xu = xg$. Thus $\varepsilon = x(gu^{-1})x^{-1}$, and we see $\operatorname{tr}(\xi(gu^{-1})) = \operatorname{tr}(\xi(\varepsilon))$ as desired. Note that twisting ε by $\tau(\phi) \in I^{\mathrm{ad}}_\phi(\mathbb{A}^p_f)$ is simply another conjugation and therefore does not change the trace. □

Now applying (4.2.7) and Lemma 4.2.8 to our starting point (4.2.1), we have

$$\operatorname{tr}(\mathbb{1}_{UgU} \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_{[\phi]} \sum_{\varepsilon} \frac{\operatorname{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q}) \backslash G^\circ_{b,\gamma\times\delta}\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})]} O^{G_b}_{\gamma\times\delta}(\mathbb{1}_{UgU}) \operatorname{tr}(\xi(\varepsilon)). \quad (4.2.9)$$

At this point the right hand side has no dependence on the function $\mathbb{1}_{UgU}$ beyond the orbital integral, so by linearity the formula applies to all acceptable functions, and we arrive at our preliminary version of the trace formula.

Proposition 4.2.10. *Let $\tau \in \Gamma(\mathcal{H})_0$ a tori-rational element satisfying Theorem 3.6.2, which we may lift to a tori-rational element of $\Gamma(\mathfrak{E}^p)_0$ (still called τ by abuse of notation). For any acceptable function $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$, we have*

$$\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_{[\phi]} \sum_{\varepsilon} \frac{\operatorname{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q}) \backslash G^\circ_{b,\gamma\times\delta}\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})]} O^{G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p)}_{\gamma\times\delta}(f) \operatorname{tr}(\xi(\varepsilon)),$$

where ϕ ranges over conjugacy classes of $\mathbf{b}$-admissible morphisms $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ and ε ranges over conjugacy classes in $I_\phi(\mathbb{Q})$, and $\gamma \times \delta$ is the image of $\operatorname{Int}(\tau(\phi))\varepsilon$ in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$.

The remainder of the section consists of technical results deferred from earlier in the section.

The following two lemmas verify the hypotheses needed to take $C = 1$ in Lemma 4.2.2 for our application above.

Lemma 4.2.11. *For U a sufficiently small compact open subgroup of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ and ϕ an admissible morphism, the stabilizer in $I_\phi(\mathbb{Q})$ of any element of $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U$ is $Z_G(\mathbb{Q}) \cap U$. The same is true if the action of $I_\phi(\mathbb{Q})$ is twisted by an element $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$.*

Proof. Let $\varepsilon \in I_\phi(\mathbb{Q})$, and suppose there is an $x \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ such that $\varepsilon x = x \bmod U$. We want to show that $\varepsilon \in Z_G(\mathbb{Q}) \cap U$. We follow the argument of [KSZ21, Lemma 5.5.3], making changes as necessary to account for the difference at p.

Regard $\varepsilon \in G(\mathbb{Q})$, and consider its image $\bar{\varepsilon} \in G^{\mathrm{ad}}(\mathbb{Q})$. We want to show that $\bar{\varepsilon}$ is trivial (so that $\varepsilon \in Z_G$). To do this, let $T \subset G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}$ be a torus containing $\bar{\varepsilon}$, which exists because $\bar{\varepsilon}$ is semi-simple, and let χ be any character of T. We start by showing $|\chi(\bar{\varepsilon})| = 1$ for any absolute value $|\cdot|$ of $\mathbb{Q}$.

In the case that $|\cdot|$ is archimedean, we regard $\bar{\varepsilon}$ as an element of $G^{\mathrm{ad}}(\mathbb{C})$, and in this case it is contained in a compact subgroup $G'(\mathbb{R})$, where G' is a compact real form of $G^{\mathrm{ad}}_{\mathbb{C}}$ ([KSZ21, Lemma 3.1.9]). The composition $|\cdot| \circ \chi$ restricted to $T(C) \cap G'(\mathbb{R})$ must be trivial, as the target $\mathbb{R}_{>}0$ has no non-trivial compact subgroups. Since $\bar{\varepsilon} \in T(C) \cap G'(\mathbb{R})$, this implies that $|\chi(\bar{\varepsilon})| = 1$.

Now consider the case that $|\cdot|$ lies above $\ell \neq p$. We are given $\varepsilon x_\ell = x_\ell \bmod U_\ell$. Thus $\varepsilon \in x_\ell U^\ell x_\ell^{-1}$, and $\bar{\varepsilon}$ regarded as an element of $G^{\mathrm{ad}}(\overline{\mathbb{Q}}_\ell)$ is contained in the compact subgroup $x_\ell \overline{U}_\ell x_\ell^{-1}$ (where $\overline{U}_\ell$ is the image of U_ℓ in $G^{\mathrm{ad}}(\mathbb{Q}_\ell)$). Again, since $\bar{\varepsilon}$ lies in a compact subgroup, we conclude $|\chi(\bar{\varepsilon})| = 1$. If the action of $I_\phi(\mathbb{Q})$ is twisted by $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, this amounts to replacing $\bar{\varepsilon}$ by a conjugate, which may change the precise compact subgroup containing it but does not affect our conclusion.

Finally the case that $|\cdot|$ lies above p. As above, we are given $\varepsilon x_p = x_p \bmod U_p$, so $\varepsilon \in x_p U_p x_p^{-1}$ in $J_b(\mathbb{Q}_p)$. Over $\overline{\mathbb{Q}}_p$ we have an embedding $J_b \hookrightarrow G$, under which $\bar{\varepsilon}$ as an element of $G(\overline{\mathbb{Q}}_p)$ is contained in the (image of the) compact subgroup $x_p U_p x_p^{-1}$. Again this implies $|\chi(\bar{\varepsilon})| = 1$.

Since we have shown $|\chi(\bar{\varepsilon})| = 1$ for every absolute value $|\cdot|$ of $\mathbb{Q}$, it follows that $\chi(\bar{\varepsilon})$ is a root of unity, and we can shrink U if necessary so that $\chi(\varepsilon) = 1$. Since we can do this for any character of T, it must be that $\bar{\varepsilon} = 1$.

This shows that $\varepsilon \in Z_G(\overline{\mathbb{Q}})$, and since $I_\phi(\mathbb{Q}) \cap Z_G(\overline{\mathbb{Q}}) = Z_G(\mathbb{Q})$, we furthermore have $\varepsilon \in Z_G(\mathbb{Q})$. Together with the fact that $x^{-1}\varepsilon x \in U$, this implies $\varepsilon \in U$, and so $\varepsilon \in Z_G(\mathbb{Q}) \cap U$ as desired. □

Lemma 4.2.12. *For a sufficiently small compact open subgroup $U \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ we have $Z_G(\mathbb{Q}) \cap U = \{1\}$.*

Proof. Since G is part of a Shimura datum of Hodge type, Z_G° satisfies the Serre condition—this is (equivalent to) the condition that Z_G° is isogenous over $\mathbb{Q}$ to a torus $T^+ \times T^-$ where T^+ is split over $\mathbb{Q}$ and T^- is compact over $\mathbb{R}$. This implies that $Z_G^\circ(\mathbb{Q})$ is discrete in $Z_G^\circ(\mathbb{A}_f)$ (e.g. [KSZ21, Lemma 2.10.3]), and via

$$Z_G^\circ(\mathbb{A}_f) \hookrightarrow G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$$

we see $Z_G^\circ(\mathbb{Q})$ is discrete in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ (the embedding $Z_G \hookrightarrow J_b$ coming from the fact that J_b is an inner form of a Levi subgroup of G). Since $[Z_G(\mathbb{Q}) : Z_G^\circ(\mathbb{Q})]$ is finite, $Z_G(\mathbb{Q})$ is also discrete in $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$. Thus any sufficiently small compact open subgroup U will intersect $Z_G(\mathbb{Q})$ trivially. □

The following lemma demonstrates that the actions of $I_{\phi,\varepsilon}(\mathbb{Q})$ and $U \cap gUg^{-1}$ on $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ do not interfere with each other, so we can "treat them separately".

Lemma 4.2.13. *For any $x \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ and sufficiently small compact open subgroup $U \subset G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$, we have*

$$I_{\phi,\varepsilon}(\mathbb{Q})x(U \cap gUg^{-1}) = \coprod_{y \in I_{\phi,\varepsilon}(\mathbb{Q})} yx(U \cap gUg^{-1}).$$

In other words, $I_{\phi,\varepsilon}(\mathbb{Q})$ acts freely on $G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/(U \cap gUg^{-1})$. The same is true if the action of $I_\phi(\mathbb{Q})$ is twisted by an element $\tau(\phi) \in I_\phi^{\text{ad}}(\mathbb{A}_f^p)$.

Proof. Note that if $y_1 x u_1 = y_2 x u_2$ for some $y_1, y_2 \in I_{\phi,\varepsilon}(\mathbb{Q})$ and $u_1, u_2 \in U \cap gUg^{-1}$, then

$$y_2^{-1} y_1 = x u_2 u_1^{-1} x^{-1} \text{ lies in } I_{\phi,\varepsilon}(\mathbb{Q}) \cap x(U \cap gUg^{-1})x^{-1}.$$

Thus it is enough to show that $I_{\phi,\varepsilon}(\mathbb{Q}) \cap x(U \cap gUg^{-1})x^{-1}$ is trivial, for that would make $y_2^{-1} y_1 = 1$ and so $y_1 = y_2$.

Recall from Lemma 4.2.11 that the stabilizer in $I_\phi(\mathbb{Q})$ of any element $x \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)/U$ is $Z_G(\mathbb{Q}) \cap U$ (regardless of twisting by $\tau(\phi) \in I_\phi^{\text{ad}}(\mathbb{A}_f^p)$), and from 4.2.12 that $Z_G(\mathbb{Q}) \cap U$ is trivial (for sufficiently small U). That is, this stabilizer is trivial. On the other hand, the same stabilizer is $I_\phi(\mathbb{Q}) \cap xUx^{-1}$, as

$$yx = x \bmod U \qquad \Longleftrightarrow \qquad y \in xUx^{-1}.$$

This shows that $I_\phi(\mathbb{Q}) \cap xUx^{-1}$ is trivial for any $x \in G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$, and so also its subgroup $I_{\phi,\varepsilon}(\mathbb{Q}) \cap x(U \cap gUg^{-1})x^{-1}$ must also be trivial. □

4.3 LR Pairs and Kottwitz Parameters

In §4.2, we applied Milne's combinatorial lemma to the result of Fujiwara's formula to express the trace of an acceptable function as a sum over pairs (ϕ, ε). We now formalize these pairs and develop the theory necessary to work with them.

Definition 4.3.1. An *LR pair* is a pair (ϕ, ε) where $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ is a morphism of Galois gerbs and $\varepsilon \in I_\phi(\mathbb{Q})$. The element ε can also be regarded as an element of $G(\overline{\mathbb{Q}})$ via $I_\phi(\mathbb{Q}) \subset G(\overline{\mathbb{Q}})$. Two LR pairs (ϕ_1, ε_1) and (ϕ_2, ε_2) are *conjugate* if there is an element $g \in G(\overline{\mathbb{Q}})$ which conjugates ϕ_1 to ϕ_2 and ε_1 to ε_2.

An LR pair (ϕ, ε) is *semi-admissible* if ϕ is admissible, and **b**-*admissible* if ϕ is **b**-admissible. We denote the set of LR pairs by $\mathcal{LRP}$, the subset of semi-admissible pairs by $\mathcal{LRP}_{\text{sa}}$, and the subset of **b**-admissible pairs by $\mathcal{LRP}_{\mathbf{b}\text{-adm}}$.

Note that if (ϕ, ε) is a semi-admissible LR pair then ε is semi-simple, as I_ϕ/Z_G is compact over $\mathbb{R}$ (cf. [KSZ21, 4.1.2]).

4.3.2 Let (ϕ, ε) be a semi-admissible LR pair. As in 2.6.6, the morphism $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate by some $g \in G(\overline{\mathbb{Q}}_p)$ to an unramified morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_G(p)$, which defines an element $b_\theta \in G(\mathbb{Q}_p^{\mathrm{ur}})$. Since ε commutes with ϕ, its conjugate $\varepsilon' = g\varepsilon g^{-1}$ commutes with θ. Then for any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$ we have

$$1 \rtimes \rho = \theta(\rho) = \mathrm{Int}(\varepsilon') \circ \theta(\rho) = \mathrm{Int}(\varepsilon')(1 \rtimes \rho) = \varepsilon'\rho(\varepsilon')^{-1} \rtimes \rho.$$

This shows that ε' is fixed by all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}})$, and is therefore an element of $G(\mathbb{Q}_p^{\mathrm{ur}})$ (a priori only being an element of $G(\overline{\mathbb{Q}}_p)$). Furthermore, since ε' commutes with θ^{ur}, it must σ-centralize b_θ, and we can regard it as an element of $J_{b_\theta}(\mathbb{Q}_p)$.

A pair (b_θ, ε') arising from (ϕ, ε) in this way is said to be a *p-adic realization* of (ϕ, ε). (For comparison, the set of p-adic realizations is called $\mathrm{cls}_p(\phi, \varepsilon)$ in [KSZ21, 4.1.4]). Note that all p-adic realizations of an LR pair are conjugate in the following sense: if $g_1, g_2 \in G(\overline{\mathbb{Q}}_p)$ give rise to p-adic realizations (b_1, ε_1) and (b_2, ε_2) respectively, then

$$b_2 = (g_2 g_1^{-1}) b_1 \sigma(g_2 g_1^{-1})^{-1}$$

and conjugation by $g_2 g_1^{-1}$ gives an isomorphism $J_{b_1} \to J_{b_2}$ sending ε_1 to ε_2.

Definition 4.3.3. Let (ϕ, ε) be a semi-admissible LR pair with (b_θ, ε') a p-adic realization. We can regard ε' as an element of $J_{b_\theta}(\mathbb{Q}_p)$. Define (ϕ, ε) to be *acceptable* if ε' is acceptable as an element of $J_{b_\theta}(\mathbb{Q}_p)$. Note that this does not depend on the choice of p-adic realization, as in the paragraph just above.

4.3.4 As we alluded to in §3, the connection between admissible morphisms and Kottwitz triples required for the Langlands-Rapoport conjecture is made using so-called "special point data". These will also play a role in the connection between LR pairs and Kottwitz parameters, so here we introduce them properly.

A *special point datum* is essentially a morphism from a zero-dimensional Shimura datum to our fixed Shimura datum (G, X); to be precise, it is a triple (T, h_T, i) where

- T is a torus,
- $h_T : \mathbb{S} \to T_{\mathbb{R}}$ is a morphism from the Deligne torus, and
- $i : T \to G$ is an embedding realizing T as a maximal torus of G defined over $\mathbb{Q}$, and sending h_T into X.

A special point datum induces an LR pair in the following way. Recall from 2.6.3 that the quasi-motivic Galois gerb is equipped with a distinguished morphism $\psi : \mathfrak{Q} \to \mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}} \mathbb{G}_m}$. Composing this with the map $\mathfrak{G}_{\mathrm{Res}_{\overline{\mathbb{Q}}/\mathbb{Q}} \mathbb{G}_m} \to \mathfrak{G}_T$ induced by the cocharacter μ_{h_T} and the map $\mathfrak{G}_T \to \mathfrak{G}_G$ induced by the inclusion i, we obtain an admissible morphism $\phi = i \circ \psi_{\mu_{h_T}} : \mathfrak{Q} \to \mathfrak{G}_G$.

Furthermore, in this setup $T(\mathbb{Q})$ as a subgroup of $G(\overline{\mathbb{Q}})$ lies inside $I_\phi(\mathbb{Q})$, so any $\varepsilon \in T(\mathbb{Q})$ makes a semi-admissible LR pair (ϕ, ε). Such a pair is called *very special*; an LR pair conjugate to a very special pair is called *special*.

It is a fact [KSZ21, 4.4.9] that every semi-admissible pair is special. This allows us to work with very special pairs whenever our application is insensitive to conjugacy, as is often the case. But very special pairs are quite restrictive, so it is useful to have another notion (contained in the following definition) which preserves some of the essential good properties of very special morphisms, but is more flexible.

Definition 4.3.5. An LR pair (ϕ, ε) is *gg* (abbreviated from *günstig gelegen*, German for "well-positioned") if

- ϕ^Δ is defined over $\mathbb{Q}$;
- ε lies in $G(\mathbb{Q})$ (under the inclusion $I_\phi(\mathbb{Q}) \subset G(\overline{\mathbb{Q}})$) and is semi-simple and elliptic in $G(\mathbb{R})$; and
- for any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, letting $q_\rho \in \mathfrak{Q}$ a lift of ρ and $\phi(q_\rho) = g_\rho \rtimes \rho$, we have $g_\rho \in G_\varepsilon^\circ$.

Very special pairs are gg, and consequently every semi-admissible LR pair is conjugate to a gg pair. We denote the set of gg LR pairs by $\mathcal{LRP}^{\mathrm{gg}} \subset \mathcal{LRP}$.

4.3.6 We now introduce Kottwitz parameters. These play the role of a group-theoretic analogue to LR pairs: both can be thought of as encoding motivic data related to an isogeny class of abelian varieties.

The following definition is inspired by [Shi09, Def 10.1] and [KSZ21, Def 4.6.1].

Definition 4.3.7. A *classical Kottwitz parameter* of *type* $\mathbf{b}$ is a triple $(\gamma_0, \gamma, \delta)$ where

- $\gamma_0 \in G(\mathbb{Q})$ is semi-simple and elliptic in $G(\mathbb{R})$,
- $\gamma = (\gamma_\ell)_\ell \in G(\mathbb{A}_f^p)$ such that γ_ℓ is stably conjugate to γ_0 in $G(\mathbb{Q}_\ell)$, and
- $\delta \in J_b(\mathbb{Q}_p)$ is acceptable, and conjugate to γ_0 in $G(\overline{\mathbb{Q}}_p)$ under the embedding $J_b(\mathbb{Q}_p) \to G(\overline{\mathbb{Q}}_p)$.

We say that $(\gamma_0, \gamma, \delta)$ and $(\gamma_0', \gamma', \delta')$ are *equivalent* if

- γ_0 is stably conjugate to γ_0' in $G(\mathbb{Q})$,
- γ is conjugate to γ' in $G(\mathbb{A}_f^p)$, and
- δ is conjugate to δ' in $J_b(\mathbb{Q}_p)$.

Intuitively, the elements γ_0, γ, δ represent the Frobenius action on rational, ℓ-adic, and crystalline cohomology respectively.

In order to handle the case that G_{der} is not simply connected, we need a (closely related but) more abstract treatment.

Definition 4.3.8 (cf. [KSZ21, Def 4.6.3])**.** A *Kottwitz parameter* is a triple $\mathfrak{c} = (\gamma_0, a, [b_0])$ where

1. $\gamma_0 \in G(\mathbb{Q})$ is semi-simple and elliptic in $G(\mathbb{R})$, and we write $I_0 = G_{\gamma_0}^\circ$;
2. a is an element of
$$\mathfrak{D}(I_0, G; \mathbb{A}_f^p) = \ker\left(H^1(I_0, \mathbb{A}_f^p) \to H^1(G, \mathbb{A}_f^p)\right);$$
3. $[b_0] \in B(I_0)$; and
4. the image of $[b_0]$ under the Kottwitz map $B(I_0) \to B(G) \xrightarrow{\kappa} \pi_1(G)_{\Gamma_p}$ is equal to the image of μ^{-1}, where μ is the cocharacter induced by an(y) element of X.

4.3.9 An isomorphism of Kottwitz parameters is essentially given by conjugation by $G(\overline{\mathbb{Q}})$. We make this precise as follows. Let $(\gamma_0, a, [b_0])$ be a Kottwitz parameter, $I_0 = G_{\gamma_0}^\circ$, and $u \in G(\overline{\mathbb{Q}})$ an element such that $\mathrm{Int}(u)\gamma_0 = \gamma_0'$ is again in $G(\mathbb{Q})$ and $u^{-1}\rho(u) \in I_0$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Write $I_0' = G_{\gamma_0'}^\circ$.

To relate the away-from-p parts, we consider the bijection

$$\begin{aligned} u_* : \mathfrak{D}(I_0, G; \mathbb{A}_f^p) &\to \mathfrak{D}(I_0', G; \mathbb{A}_f^p) \\ e_\rho &\mapsto u e_\rho \rho(u)^{-1} \end{aligned}$$

induced by the element u.

To relate the p-parts, we construct a bijection $u_* : B(I_0) \to B(I_0')$. The cocycle $\rho \mapsto u^{-1}\rho(u) \in Z^1(\mathbb{Q}_p, I_0)$ is trivial in $H^1(\breve{\mathbb{Q}}_p, I_0)$ by the Steinberg vanishing theorem. That is, we can find $d \in I_0(\breve{\overline{\mathbb{Q}}}_p)$ so that $u^{-1}\rho(u) = d^{-1}\rho(d)$ for all

$\rho \in \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/\breve{\mathbb{Q}}_p)$. Then we have $ud^{-1} = \rho(ud^{-1})$ for all such ρ, so $u_0 := ud^{-1}$ lies in $G(\breve{\mathbb{Q}}_p)$.

Since d commutes with γ_0, we have $u_0\gamma_0 u_0^{-1} = \gamma_0'$, and thus u_0 induces a bijection

$$\begin{aligned} u_* : B(I_0) &\to B(I_0') \\ [b] &\mapsto [u_0 b\sigma(u_0)^{-1}]. \end{aligned}$$

This bijection is independent of d (and therefore deserves the name u_*) because any other choice of d is related by an element of $I_0(\breve{\mathbb{Q}}_p)$ and therefore its σ-conjugation of b does not change the class in $B(I_0)$.

Now with the above setup, an *isomorphism* of Kottwitz parameters $(\gamma_0, a, [b_0])$ and $(\gamma_0', a', [b_0'])$ is an element $u \in G(\overline{\mathbb{Q}})$ such that

- $\mathrm{Int}(u)\gamma_0 = \gamma_0'$ and $u^{-1}\rho(u) \in I_0$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (i.e. u stably conjugates γ_0 to γ_0'),
- the bijection $u_* : \mathfrak{D}(I_0, G; \mathbb{A}_f^p) \to \mathfrak{D}(I_0', G; \mathbb{A}_f^p)$ sends a to a', and
- the bijection $u_* : B(I_0) \to B(I_0')$ sends $[b_0]$ to $[b_0']$.

4.3.10 We define notions of acceptable and $\mathbf{b}$-admissible Kottwitz parameters, which we will see correspond precisely with acceptable and $\mathbf{b}$-admissible LR pairs.

A Kottwitz parameter $(\gamma_0, a, [b_0])$ is $\mathbf{b}$-*admissible* if the map $B(I_0) \to B(G)$ sends $[b_0]$ to $\mathbf{b}$. Note that we have fixed $\mathbf{b}$ in $B(G, \mu^{-1})$, so a $\mathbf{b}$-admissible Kottwitz parameter automatically satisfies item 4 of Definition 4.3.8.

To define acceptable, let $(\gamma_0, a, [b_0])$ be a Kottwitz parameter. Since b_0 (any representative of $[b_0]$) lies in $I_0 = G_{\gamma_0}^\circ$, we see that γ_0 centralizes b_0; since γ_0 is rational, we see that γ_0 furthermore σ-centralizes b_0, so we can regard γ_0 as an element of $J_{b_0}(\mathbb{Q}_p)$.

Say that $(\gamma_0, a, [b_0])$ is *acceptable* if γ_0 is acceptable as an element of $J_{b_0}(\mathbb{Q}_p)$. This does not depend on the choice of representative b_0 of $[b_0]$, as a different representative b_0'—being σ-conjugate to b_0—will admit an isomorphism $J_{b_0} \xrightarrow{\sim} J_{b_0'}$ sending γ_0 to a conjugate, and acceptability is insensitive to conjugation.

4.3.11 A key ingredient for relating Kottwitz parameters to LR pairs is the Kottwitz invariant, which is a cohomological measure of the global compatibility between the local pieces of the Kottwitz parameter. To be precise, we will see that in order to be related to an LR pair, a Kottwitz parameter must have invariant zero.

We define the Kottwitz invariant exactly as in [KSZ21, 4.7]. The full details can be found in *loc. cit.*, but for convenience we repeat the main ideas of the construction here.

Let $\mathfrak{c} = (\gamma_0, a, [b_0])$ be a Kottwitz parameter, $I_0 = G^{\circ}_{\gamma_0}$, and consider the group

$$\mathfrak{E}(\mathfrak{c}) = \mathfrak{E}(I_0, G; \mathbb{A}/\mathbb{Q}) = \operatorname{coker}\left(H^0_{\mathrm{ab}}(\mathbb{A}, G) \to H^0_{\mathrm{ab}}(\mathbb{A}/\mathbb{Q}, I_0 \to G)\right)$$

where H^0_{ab} is the abelianized Galois cohomology of [Lab99]. The Kottwitz invariant of the Kottwitz parameter $\mathfrak{c}$ will be an element $\alpha(\mathfrak{c}) \in \mathfrak{E}(\mathfrak{c})$. The group $\mathfrak{E}(\mathfrak{c})$ also has the following useful description; let

$$K(\mathfrak{c}) = \ker\left(\pi_1(I_0) \to \pi_1(G)\right)$$

and write $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\Gamma_v = \mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$.

Lemma 4.3.12 ([KSZ21, Cor. 1.3.6])**.** *There is a canonical isomorphism*

$$\mathfrak{E}(\mathfrak{c}) \xrightarrow{\sim} \frac{K(\mathfrak{c})_{\Gamma,\mathrm{tors}}}{\bigoplus_v \ker(K(\mathfrak{c})_{\Gamma_v,\mathrm{tors}} \to \pi_1(I_0)_{\Gamma_v})}$$

where v runs through all places of $\mathbb{Q}$.

Furthermore by [KSZ21, Lemma 1.3.5] we have $K(\mathfrak{c})_\Gamma = K(\mathfrak{c})_{\Gamma,\mathrm{tors}}$. We will produce the Kottwitz invariant $\alpha(\mathfrak{c}) \in \mathfrak{E}(\mathfrak{c})$ by constructing an element of $\tilde{\beta}(\mathfrak{c}) \in K(\mathfrak{c})$ and using the above isomorphism to take its image in $\mathfrak{E}(\mathfrak{c})$ (via the quotient to $K(\mathfrak{c})_\Gamma$).

We begin with the local components at $\ell \neq p, \infty$. Recall the set $\mathfrak{D}(I_0, G; \mathbb{A}^p_f)$ and define its abelianized analogue $\mathfrak{E}(I_0, G; \mathbb{A}^p_f)$ by

$$\mathfrak{D}(I_0, G; \mathbb{A}^p_f) = \ker\left(H^1(\mathbb{A}^p_f, I_0) \to H^1(\mathbb{A}^p_f, G)\right),$$
$$\mathfrak{E}(I_0, G; \mathbb{A}^p_f) = \ker\left(H^1_{\mathrm{ab}}(\mathbb{A}^p_f, I_0) \to H^1_{\mathrm{ab}}(\mathbb{A}^p_f, G)\right).$$

The abelianization map $\mathrm{ab}^0 : H^0(*,*) \to H^0_{\mathrm{ab}}(*,*)$ induces an isomorphism (i.e. bijection of pointed sets) $\mathfrak{D}(I_0, G; \mathbb{A}^p_f) \xrightarrow{\sim} \mathfrak{E}(I_0, G; \mathbb{A}^p_f)$.

As in [KSZ21, 4.7.1] we have a canonical isomorphism $H^1_{\mathrm{ab}}(\mathbb{A}^p_f, I_0) \cong \bigoplus_{\ell \neq p,\infty} \pi_1(I_0)_{\Gamma_\ell,\mathrm{tors}}$. Thus we can write a as $(\beta_\ell)_{\ell \neq p,\infty}$ via

$$\begin{array}{ccccccc} \mathfrak{D}(I_0, G; \mathbb{A}^p_f) & \xrightarrow{\sim} & \mathfrak{E}(I_0, G; \mathbb{A}^p_f) & \hookrightarrow & H^1_{\mathrm{ab}}(\mathbb{A}^p_f, I_0) & \xrightarrow{\sim} & \bigoplus_{\ell \neq p,\infty} \pi_1(I_0)_{\Gamma_v,\mathrm{tors}} \\ a & & \longmapsto & & & & (\beta_\ell)_{\ell \neq p,\infty} \end{array}$$

We can choose lifts $\tilde{\beta}_\ell \in \pi_1(I_0)$ of β_ℓ such that $\tilde{\beta}_\ell$ maps to zero in $\pi_1(G)$ (thus $\tilde{\beta}_\ell \in K(\mathfrak{c})$), and $\tilde{\beta}_\ell = 0$ for almost all ℓ.

Now the local component at p. Let $\beta_p = \kappa_{I_0}([b_0]) \in \pi_1(I_0)_{\Gamma_p}$. We can choose a lift $\tilde{\beta}_p \in \pi_1(I_0)$ mapping to $-[\mu] \in \pi_1(G)$, where $[\mu]$ is the image of a(ny) cocharacter μ induced by an element of X.

Finally the local component at ∞. Let $T \subset G_{\mathbb{R}}$ an elliptic maximal torus containing γ_0. Then $T \subset I_{0,\mathbb{R}}$. Since T is elliptic, we can choose an element $h \in X$ factoring through T. Define β_∞ to be the image of $\mu_h \in X_*(T)$ in $\pi_1(I_0)_{\Gamma_\infty}$. We can choose a lift $\tilde{\beta}_\infty \in \pi_1(I_0)$ mapping to $[\mu]$ in $\pi_1(G)$.

By construction we have $\tilde{\beta}_\ell \in K(\mathfrak{c})$ for all $\ell \neq p, \infty$, with $\tilde{\beta}_\ell = 0$ for all but finitely many ℓ, and $\tilde{\beta}_p + \tilde{\beta}_\infty \in K(\mathfrak{c})$. Thus we can define

$$\tilde{\beta}(\mathfrak{c}) = \sum_v \tilde{\beta}_v \in K(\mathfrak{c})$$

where v runs over all places of $\mathbb{Q}$. We define the *Kottwitz invariant* $\alpha(\mathfrak{c}) \in \mathfrak{E}(\mathfrak{c})$ of the Kottwitz parameter $\mathfrak{c}$ to be the image of the element $\tilde{\beta}(\mathfrak{c})$ in $\mathfrak{E}(\mathfrak{c})$.

4.3.13 Recall that we left off our trace formula in Proposition 4.2.10 as a sum over $\mathbf{b}$-admissible LR pairs. For applications, we want our final trace formula to be a sum over Kottwitz parameters. To start the translation, we now define a map from LR pairs to Kottwitz parameters, following [KSZ21, 4.8.1].

Let (ϕ, ε) be a semi-admissible LR pair, and $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$. After conjugation by an element of $G(\overline{\mathbb{A}}_f^p)$ we may assume that (ϕ, ε) is gg—we verify in Lemma 4.3.15 that the result is insensitive to this conjugation. We will define a Kottwitz parameter $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ associated to (ϕ, ε) and $\tau(\phi)$.

Define $\gamma_0 = \varepsilon$. By the gg condition, ε is contained in $G(\mathbb{Q})$ and is semi-simple and elliptic in $G(\mathbb{R})$, verifying the requirements for γ_0 in Definition 4.3.8. We write $I_0 = G_{\gamma_0}^\circ = G_\varepsilon^\circ$.

Next we consider a. Recall the cocycles $\zeta_\phi^{p,\infty}$ and $\zeta_{\phi,\ell}$ of 2.6.12. The gg condition

$$\phi(q_\rho) = g_\rho \rtimes \rho \text{ has } g_\rho \in G_\varepsilon^\circ = I_0 \text{ for } q_\rho \text{ any lift of } \rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \qquad (4.3.14)$$

implies that $\zeta_\phi^{p,\infty}$ is valued in $I_0(\overline{\mathbb{A}}_f^p)$.

Choose a lift $\tilde{\tau} \in I_\phi(\overline{\mathbb{A}}_f^p)$ of $\tau(\phi)$, and define a cocycle $A : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to I_0(\overline{\mathbb{A}}_f^p)$ by

$$A(\rho) = t_\rho \zeta_\phi^{p,\infty}(\rho)$$

where $t_\rho = \tilde{\tau}^{-1}\rho(\tilde{\tau}) \in Z_{I_\phi}(\overline{\mathbb{A}}_f^p)$, acting by ρ via the $\mathbb{Q}$-structure of I_ϕ—we can regard t_ρ as an element of $I_0(\overline{\mathbb{A}}_f^p)$ because the natural embedding $Z_{I_\phi} \to G$ factors through I_0.

The cocycle A splits over $G(\overline{\mathbb{A}}_f^p)$. To see this, we write $\tilde{\tau}_G$ for the image of $\tilde{\tau}$ in G—we distinguish these because $\tilde{\tau}$ is subject to the Galois action given by the $\mathbb{Q}$-structure on I_0, while $\tilde{\tau}_G$ is subject to that given by the $\mathbb{Q}$-structure on G. With respect to the Galois action on G, the element $\rho(\tilde{\tau})$ becomes $\zeta_\phi^{p,\infty}(\rho)\rho(\tilde{\tau}_G)\zeta_\phi^{p,\infty}(\rho)^{-1}$, and so

$$A(\rho) = \tilde{\tau}^{-1}\rho(\tilde{\tau})\zeta_\phi^{p,\infty}(\rho) = \tilde{\tau}_G^{-1}\zeta_\phi^{p,\infty}(\rho)\rho(\tilde{\tau}_G)\zeta_\phi^{p,\infty}(\rho)^{-1}\zeta_\phi^{p,\infty}(\rho) = \tilde{\tau}_G^{-1}\zeta_\phi^{p,\infty}(\rho)\rho(\tilde{\tau}_G).$$

Combined with the fact that $\zeta_\phi^{p,\infty}$ splits in $G(\overline{\mathbb{A}}_f^p)$ (realized as $\rho \mapsto x\rho(x)^{-1}$ for $x \in X^p(\phi)$), this shows that A splits in $G(\overline{\mathbb{A}}_f^p)$ as well.

We define $a \in \mathfrak{D}(I_0, G; \mathbb{A}_f^p)$ in our Kottwitz parameter to be the class defined by the image of A. This does not depend on the choice of lift $\tilde{\tau}$, because two choices differ by an element of $Z_{I_\phi}(\overline{\mathbb{A}}_f^p)$ which commutes with $\zeta_\phi^{p,\infty}$.

Finally we construct $[b_0]$. The same gg condition (4.3.14) above implies that ϕ factors

$$\phi : \mathfrak{Q} \xrightarrow{\phi_0} \mathfrak{G}_{I_0} \to \mathfrak{G}_G.$$

Then (ϕ_0, ε) is again an LR pair, and taking a p-adic realization (b_θ, ε') produces a class $[b_0] = [b_\theta] \in B(I_0)$ (independent of the choice of p-adic realization, as different choices of b_θ will still be σ-conjugate in I_0). This finishes the construction of $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$.

Note that we have taken $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, but by [KSZ21, Prop 4.8.2] this construction only depends on its image in $\mathfrak{E}^p(\phi) = I_\phi(\mathbb{A}_f^p)\backslash I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, so we have a well-defined Kottwitz parameter $\mathbf{t}(\phi, \varepsilon, \tau(\phi)$ associated to a semi-admissible pair (ϕ, ε) and an element $\tau \in \Gamma(\mathfrak{E}^p)$.

We want to show that this construction only depends on the conjugacy class of the LR pair. In particular, after conjugating we have worked in the case that our LR pair is gg, so we want to show that if two gg pairs are conjugate then the resulting Kottwitz parameters are isomorphic. For this we need to assume that the τ-twists are well-behaved under conjugation as well.

Lemma 4.3.15. *Let (ϕ, ε) and (ϕ', ε') be gg LR pairs, and $\tau \in \Gamma(\mathfrak{E}^p)_1$. Write $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ and $\mathbf{t}(\phi', \varepsilon', \tau(\phi')) = (\gamma_0', a', [b_0'])$ for the associated Kottwitz parameters. If $u \in G(\overline{\mathbb{Q}})$ conjugates (ϕ, ε) to (ϕ', ε'), then*

1. *$u\rho(u)^{-1} \in G_{\varepsilon'}^\circ$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and*
2. *u gives an isomorphism $(\gamma_0, a, [b_0]) \cong (\gamma_0', a', [b_0'])$.*

The argument is essentially similar to [KSZ21, Prop 4.8.3], but for convenience we repeat it here.

Proof. We begin with the first claim. Let $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $q_\rho \in \mathfrak{Q}$ a lift of ρ. Writing $\phi(q_\rho) = g_\rho \rtimes \rho$, we have

$$\phi'(q_\rho) = u g_\rho \rho(u)^{-1} \rtimes \rho.$$

By the gg property of (ϕ, ε) we have $g_\rho \in G_\varepsilon^\circ$, and combined with $\mathrm{Int}(u)\varepsilon = \varepsilon'$ this implies $u g_\rho u^{-1} \in G_{\varepsilon'}^\circ$. On the other hand, the gg property of (ϕ', ε') implies $u g_\rho \rho(u)^{-1} \in G_{\varepsilon'}^\circ$. Combining these, we find

$$u\rho(u)^{-1} = \left(u g_\rho u^{-1}\right)^{-1} \cdot u g_\rho \rho(u)^{-1} \in G_{\varepsilon'}^\circ$$

as desired.

Now for the second claim. We will verify the three conditions for an isomorphism of Kottwitz parameters in 4.3.9. By construction $\gamma_0 = \varepsilon$ and $\gamma_0' = \varepsilon'$, so the fact that $\mathrm{Int}(u)\varepsilon = \varepsilon'$ and $u\rho(u)^{-1} \in G_{\varepsilon'}^\circ$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ verifies the first condition.

For the away-from-p part, recall the class a is defined by the (image of the) cocycle

$$A(\rho) = \tilde{\tau}_G^{-1} \zeta_\phi^{p,\infty}(\rho)\rho(\tilde{\tau}_G),$$

where $\tilde{\tau}$ is a lift of $\tau(\phi)$ to $I_\phi(\overline{\mathbb{A}}_f^p)$ and $\tilde{\tau}_G$ is its image in $G(\overline{\mathbb{A}}_f^p)$. We will write $A'(\rho) = \tilde{\tau}_G'^{-1} \zeta_{\phi'}^{p,\infty} \rho(\tilde{\tau}_G')$ for the cocycle defining a'.

The fact that $\mathrm{Int}(u) \circ \phi = \phi'$ implies

$$u \zeta_\phi^{p,\infty} \rho(u)^{-1} = \zeta_{\phi'}^{p,\infty}.$$

We have assumed that τ lies in the subspace $\Gamma(\mathcal{H})_1$ of $\Gamma(\mathcal{H})$ which descends to admissible morphisms mod conjugacy, so $\tau(\phi)$ and $\tau(\phi')$ are related under the bijection $\mathrm{Comp}_{\phi,\phi'} : \mathcal{H}(\phi) \xrightarrow{\sim} \mathcal{H}(\phi')$. From the construction of $\mathrm{Comp}_{\phi,\phi'}$ in [KSZ21, 3.2.6] we see that

$$u \tilde{\tau}_G u^{-1} = \tilde{\tau}_G'.$$

Combining these we find

$$\begin{aligned} A'(\rho) = \tilde{\tau}_G'^{-1} \zeta_{\phi'}^{p,\infty} \rho(\tilde{\tau}_G') &= (u\tilde{\tau}_G u^{-1})^{-1} u \zeta_\phi^{p,\infty} \rho(u)^{-1} \rho(u \tilde{\tau}_G u^{-1}) \\ &= u \tilde{\tau}_G^{-1} \zeta_\phi^{p,\infty} \rho(\tilde{\tau}_G) \rho(u)^{-1} = u A(\rho) \rho(u)^{-1} \end{aligned}$$

which shows that the map $u_* : \mathfrak{D}(I_0, G; \mathbb{A}_f^p) \to \mathfrak{D}(I_0', G; \mathbb{A}_f^p)$ sends a to a', verifying the second condition.

For the p-part, we take $d \in I_0(\breve{\mathbb{Q}}_p)$ and $u_0 = ud^{-1} \in G(\breve{\mathbb{Q}}_p)$ as in 4.3.9. Recall that the bijection $u_* : B(I_0) \to B(I_0')$ sends $[b_0]$ to $[u_0 b_0 \sigma(u_0)^{-1}]$.

Consider the factorizations

$$\phi : \mathfrak{Q} \xrightarrow{\phi_0} \mathfrak{G}_{I_0} \to \mathfrak{G}, \qquad \phi' : \mathfrak{Q} \xrightarrow{\phi_0'} \mathfrak{G}_{I_0'} \to \mathfrak{G},$$

and choose $g \in I_0(\overline{\mathbb{Q}}_p)$ (resp. $g' \in I_0'(\overline{\mathbb{Q}}_p)$) such that $\theta = \mathrm{Int}(g) \circ \phi_0(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_{I_0}(p)$ is unramified (resp. $\theta' = \mathrm{Int}(g') \circ \phi_0'(p) \circ \zeta_p$ is unramified). Then $[b_0]$ is the class associated to θ and $[b_0']$ to θ'.

By construction we have $\theta' = \mathrm{Int}(g'ug^{-1})\theta$, and thus $[b_0'] = [(g'ug^{-1})b_0\sigma(g'ug^{-1})^{-1}]$. We want to identify this class with $[u_0b_0\sigma(u_0)^{-1}]$, for which it suffices to show that the element $u_0gu^{-1}g'^{-1}$ conjugating the first to the second lies in $I_0'(\breve{\mathbb{Q}}_p)$.

Note that $d \in I_0(\overline{\breve{\mathbb{Q}}}_p)$ and $g \in I_0(\overline{\mathbb{Q}}_p)$. Because $\mathrm{Int}(u)$ takes ϕ to ϕ' and therefore I_0 to I_0', we have $u_0gu^{-1} = \mathrm{Int}(u)(d^{-1}g) \in I_0'(\overline{\breve{\mathbb{Q}}}_p)$. Of course $g' \in I_0'(\overline{\mathbb{Q}}_p)$ also, so $u_0gu^{-1}g'^{-1} \in I_0(\overline{\breve{\mathbb{Q}}}_p)$.

On the other hand, $gu^{-1}g'^{-1}$ conjugates one unramified morphism to another and therefore lies in $G(\breve{\mathbb{Q}}_p)$, and by construction we also have $u_0 \in G(\breve{\mathbb{Q}}_p)$. Thus $u_0gu^{-1}g'^{-1} \in G(\breve{\mathbb{Q}}_p)$. Combining this with the above paragraph, we find $u_0gu^{-1}g'^{-1} \in I_0'(\breve{\mathbb{Q}}_p)$ as desired. □

Thus for any $\tau \in \Gamma(\mathfrak{E}^p)_1$ we have a well-defined map

$$\begin{aligned} \mathbf{t} : \mathcal{LRP}_{\mathrm{sa}}/\mathrm{conj.} &\to \mathcal{KP}/\mathrm{isom.} \\ (\phi, \varepsilon) &\mapsto \mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0]). \end{aligned}$$

In order to translate our point-counting formula into the language of Kottwitz parameters, we need a detailed understanding of this map, which is undertaken in §4.4.

Before that, we establish some basic compatibility results. The following lemma establishes the compatibility of the map with our definitions of acceptable and $\mathbf{b}$-admissible.

Lemma 4.3.16. *Let (ϕ, ε) a semi-admissible pair, $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, and $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ the associated Kottwitz parameter.*

1. *$(\gamma_0, a, [b_0])$ is $\mathbf{b}$-admissible if and only if (ϕ, ε) is $\mathbf{b}$-admissible.*

2. *$(\gamma_0, a, [b_0])$ is acceptable if and only if (ϕ, ε) is acceptable.*

Proof. For the first claim, recall that a semi-admissible pair (ϕ, ε) is $\mathbf{b}$-admissible if $\phi(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate to an unramified morphism θ with $b_\theta \in \mathbf{b}$. On the other hand, $[b_0]$ is the class (in $B(I_0)$) defined by precisely such a b_θ, and $(\gamma_0, a, [b_0])$ is defined to be $\mathbf{b}$-admissible if $[b_0]$ maps to the class $\mathbf{b}$ under $B(I_0) \to B(G)$. These conditions are manifestly equivalent.

For the second claim, by conjugation we may assume that (ϕ, ε) is gg (as acceptability only depends on the conjugacy or isomorphism class). Recall that (ϕ, ε) is acceptable if for one (hence any) p-adic realization (b_θ, ε') we have ε' acceptable in $J_{b_\theta}(\mathbb{Q}_p)$; and that $(\gamma_0, a, [b_0])$ is acceptable if γ_0 is acceptable in $J_{b_0}(\mathbb{Q}_p)$. Thus it suffices to show that (b_0, γ_0) is a p-adic realization of (ϕ, ε). This follows from the fact that b_0 is defined by an $I_0(\overline{\mathbb{Q}}_p)$-conjugate of $\phi(p) \circ \zeta_p$, which commutes with $\gamma_0 = \varepsilon$. □

4.3.17 In order to work with the orbital integrals appearing in our formula, we also need to relate LR pairs and Kottwitz parameters via classical Kottwitz parameters.

We start by associating a classical Kottwitz parameter $(\gamma_0, \gamma, \delta)$ to an LR pair. Let (ϕ, ε) be a gg acceptable $\mathbf{b}$-admissible LR pair and $\tau(\phi) \in I_\phi(\overline{\mathbb{A}}_f^p)$. By the gg condition, ε lies in $G(\mathbb{Q})$, so we can let $\gamma_0 = \varepsilon$. Then we define γ and δ as the image of $\mathrm{Int}(\tau(\phi))\varepsilon$ in $G(\mathbb{A}_f^p)$ and $J_b(\mathbb{Q}_p)$ respectively.

We check that $(\gamma_0, \gamma, \delta)$ indeed forms a classical Kottwitz parameter. The gg condition implies that γ_0 is semi-simple and elliptic in $G(\mathbb{R})$, verifying the first condition of Definition 4.3.7. We have γ_0 conjugate to γ in $G(\mathbb{A}_f^p)$ by (the image of) $\tau(\phi)$, and since $\tau^{-1}\rho(\tau) \in Z_{I_\phi} \hookrightarrow I_0 = G_{\gamma_0}^\circ$ for any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, they are stably conjugate. Since $\tau(\phi)$ is trivial at p, it is trivial that γ_0 is stably conjugate to δ. Finally, δ is acceptable because (ϕ, ε) is acceptable.

4.3.18 Next we associate a classical Kottwitz parameter to a Kottwitz parameter. Let $(\gamma_0, a, [b_0])$ be an acceptable $\mathbf{b}$-admissible Kottwitz parameter. We let $\gamma_0 = \gamma_0$, i.e. the element γ_0 of our classical Kottwitz parameter we choose to be the element γ_0 of our Kottwitz parameter. The class a determines a conjugacy class in $G(\mathbb{A}_f^p)$ stably conjugate to γ_0, and we choose γ to be an arbitrary element of this class. Finally, as our Kottwitz parameter is acceptable and $\mathbf{b}$-admissible, γ_0 can be considered as an element of $J_b(\mathbb{Q}_p)$, and we set $\delta = \gamma_0$.

As it came from a Kottwitz parameter, γ_0 is semi-simple and elliptic in $G(\mathbb{R})$. By construction γ_0 is stably conjugate to γ, and δ is acceptable and conjugate to γ_0.

The next lemma states that an LR pair and its associated Kottwitz parameter give rise to the same classical Kottwitz parameter. Recall the equivalence between classical Kottwitz parameters defined in 4.3.7.

Lemma 4.3.19. *Let (ϕ, ε) a gg acceptable $\mathbf{b}$-admissible pair, $\tau(\phi) \in I_\phi(\overline{\mathbb{A}}_f^p)$, and $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ the associated Kottwitz parameter. Let $(\gamma_0, \gamma, \delta)$ be the classical Kottwitz parameter associated to $(\gamma_0, a, [b_0])$, and $(\gamma_0', \gamma', \delta')$ the classical Kottwitz parameter associated to (ϕ, ε) and $\tau(\phi)$. Then $(\gamma_0, \gamma, \delta)$ and $(\gamma_0', \gamma', \delta')$ are equivalent.*

Proof. By construction we have $\gamma_0' = \varepsilon = \gamma_0$—in particular γ_0 is stably conjugate to γ_0', which verifies the first condition of equivalence. Examining the cocycle

$$A(\rho) = \tau(\phi)^{-1}\rho(\tau(\phi))\zeta_\phi^{p,\infty}(\rho)$$

defining a, and noting that $\zeta_\phi^{p,\infty}$ commutes with $\varepsilon = \gamma_0$, we see that the conjugacy class of γ is the conjugacy class of $\mathrm{Int}(\tau(\phi))$, which shows that γ and γ' are conjugate in $G(\mathbb{A}_f^p)$. By construction, δ and δ' are conjugate as well. □

4.4 The Image of the Map $\mathcal{LRP} \to \mathcal{KP}$

Recall from §4.3 we have defined a map $\mathbf{t} : \mathcal{LRP}_{\mathrm{sa}}/\mathrm{conj.} \to \mathcal{KP}/\mathrm{isom.}$ for any $\tau \in \Gamma(\mathfrak{E}^p)_1$. The main result of this section is that the image of the set of acceptable **b**-admissible LR pairs is the set of acceptable **b**-admissible Kottwitz parameters with trivial Kottwitz invariant. This will allow us later to rephrase our trace formula from its parametrization by LR pairs to a parametrization by Kottwitz parameters.

We begin by observing that the image of acceptable **b**-admissible LR pairs lies in the set of acceptable **b**-admissible Kottwitz parameters with trivial Kottwitz invariant. Let (ϕ, ε) an acceptable **b**-admissible LR pair and $\tau \in \Gamma(\mathfrak{E}^p)$ a tori-rational element. By Lemma 4.3.16, its image $\mathbf{t}(\phi, \varepsilon, \tau(\phi))$ is also acceptable and **b**-admissible. The following lemma tells us that its image also has trivial Kottwitz invariant.

Lemma 4.4.1 ([KSZ21, Prop. 5.2.6])**.** *Let (ϕ, ε) be a semi-admissible LR pair, $\tau \in \Gamma(\mathfrak{E}^p)_1$ a tori-rational element, and $\mathfrak{c} = \mathbf{t}(\phi, \varepsilon, \tau(\phi))$ the associated Kottwitz parameter. Then the Kottwitz invariant $\alpha(\mathfrak{c})$ is zero.*

4.4.2 Now we want to show that given an acceptable **b**-admissible Kottwitz parameter $(\gamma_0, a, [b_0])$ and $\tau \in \Gamma(\mathfrak{E}^p)_0$ tori-rational, we can find an acceptable **b**-admissible LR pair (ϕ, ε) such that $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$.

If μ is a cocharacter of an algebraic group H, over $\mathbb{Q}_p$, we define $[b_{\mathrm{bas}}(\mu)] \in B(H)$ to be the unique basic class in $B(H, \mu)$.

Lemma 4.4.3. *Let $(\gamma_0, a, [b_0])$ be an acceptable **b**-admissible Kottwitz parameter, and write $I_0 = G^\circ_{\gamma_0}$. Then there exists a maximal torus $T \subset I_0$ over $\mathbb{Q}$ and $x \in X$ such that h_x factors through $T_\mathbb{R}$ (so $\mu_x \in X_*(T)$) and $[b_{\mathrm{bas}}(\mu_x^{-1})] \in B(T)$ maps to $[b_0]$ in $B(I_0)$.*

This lemma and its proof are directly inspired from [KMS, Prop 1.2.5].

Proof. Since γ_0 is acceptable with respect to $[b_0]$, we have $I_0 \subset M_{b_0}$ by Lemma 2.2.5. This implies that I_0 centralizes the slope homomorphism of $[b_0]$, i.e. $[b_0]$ is basic in $B(I_0)$. Let $J_{b_0}^{I_0}$ be the inner form of I_0 defined by $[b_0]$.

Recall we can consider γ_0 as an element of $J_{b_0}(\mathbb{Q}_p)$. The group $J_{b_0}^{I_0}$ is the connected centralizer of γ_0 in J_{b_0}. Choose an elliptic maximal torus $T' \subset J_{b_0}$ containing γ_0. Then T' is contained in $J_{b_0}^{I_0}$, and being elliptic, transfers to a torus $T' \xrightarrow{\sim} T_p \subset I_0$ over $\mathbb{Q}_p$.

By [KMS, Cor 1.1.17], there is a representative $\mu_p \in X_*(T_p)$ of $\{\mu_X\}$ (the conjugacy class of cocharacters induced by the Shimura datum (G, X)) such that $[b_{\mathrm{bas}}(\mu_p^{-1})] \in B(T_p)$ maps to $[b_0]$ in $B(I_0)$ (note that hypothesis (1.1.3.1) of that Corollary is satisfied because $[b_0]$ is basic in $B(I_0)$, and $[b_0]$ is μ^{-1}-admissible because it maps to $\mathbf{b} \in B(G, \mu^{-1})$).

Choose a maximal torus $T_\infty \subset G_\mathbb{R}$ containing γ_0, so in particular $T_\infty \subset I_{0,\mathbb{R}}$. Since every element of X factors through an elliptic maximal torus of $G_\mathbb{R}$ and all such tori are $G(\mathbb{R})$-conjugate, we can choose $x \in X$ such that h_x factors through T_∞.

Having produced $T_p \subset I_{0,\mathbb{Q}_p}$ and $T_\infty \subset I_{0,\mathbb{R}}$ with the necessary properties, the rest of the proof can be read verbatim from the proof of [KMS, Prop 1.2.5], replacing their G by our I_0. □

Proposition 4.4.4. *Let $(\gamma_0, a, [b_0])$ be an acceptable* $\mathbf{b}$*-admissible Kottwitz parameter. Then there is an admissible morphism ϕ_0 which forms a gg acceptable* $\mathbf{b}$*-admissible LR pair (ϕ_0, γ_0).*

Proof. By Lemma 4.4.3, there exists a maximal torus $T \subset I_0$ over $\mathbb{Q}$ and $x \in X$ such that h_x factors through $T_\mathbb{R}$ and $[b_{\mathrm{bas}}(\mu_x^{-1})] \in B(T)$ maps to $[b_0] \in B(I_0)$. In particular, $(T, h = h_x)$ forms a special point datum.

Let $\phi_0 = i \circ \psi_{\mu_h}$ be the admissible morphism induced from the special point datum (T, h) as in 4.3.4. Since $T \subset I_0$ is maximal and γ_0 is central in I_0, we have $\gamma_0 \in T$, so we can form the LR pair (ϕ_0, γ_0). By construction it is very special and in particular gg. It remains to show that this LR pair is acceptable and $\mathbf{b}$-admissible.

We check $\mathbf{b}$-admissibility first. Consider $\psi_{\mu_h} : \mathfrak{Q} \to \mathfrak{G}_T$ and its p-part $\psi_{\mu_h}(p) \circ \zeta_p : \mathfrak{G}_p \to \mathfrak{G}_T(p)$. As in 2.6.6, the latter is conjugate by some element $y \in T(\overline{\mathbb{Q}}_p)$ to an unramified morphism $\theta : \mathfrak{G}_p \to \mathfrak{G}_T(p)$ which gives rise to an element $[b_\theta] \in B(T)$.

By [KSZ21, Lemma 2.7.2], the image $\kappa_T(b_\theta) \in X_*(T)_{\Gamma_p}$ of $[b_\theta]$ under the Kottwitz map is equal to the image of $\mu_h^{-1} \in X_*(T)$. Since the Kottwitz map is a bijection in the case of a torus, we conclude that $[b_\theta] = [b_{\mathrm{bas}}(\mu_h^{-1})]$ in $B(T)$, as the latter also maps to μ_h^{-1}. By our conclusions from Lemma 4.4.3 above we find that $[b_\theta] \in B(T)$ maps to $[b_0] \in B(I_0)$ and therefore to $\mathbf{b} \in B(G)$ (by $\mathbf{b}$-admissibility of our Kottwitz parameter).

Now conjugating $\phi_0(p) \circ \zeta_p = i \circ \psi_{\mu_h}(p) \circ \zeta_p$ by the same element $y \in T(\overline{\mathbb{Q}}_p)$ produces the unramified morphism $i \circ \theta$, and the associated class $[b_{i\circ\theta}] \in B(G)$ is the image of the class $[b_\theta] \in B(T)$. Thus we have $[b_{i\circ\theta}] = \mathbf{b}$, and this verifies that (ϕ_0, γ_0) is $\mathbf{b}$-admissible.

Now we check acceptability. Note that we conjugated by an element $y \in T(\overline{\mathbb{Q}}_p)$ to produce the unramified morphism $i \circ \theta$. In particular y commutes with γ_0, so $(b_{i\circ\theta}, \gamma_0)$ is a p-adic realization of (ϕ_0, γ_0). We have assumed that γ_0 is acceptable with respect to $[b_0]$, and shown that $[b_0] = [b_\theta]$ in $B(I_0)$. This shows that γ_0 is acceptable with respect to $[b_{i\circ\theta}]$, and (ϕ_0, γ_0) is acceptable. □

4.4.5 Starting with an acceptable $\mathbf{b}$-admissible Kottwitz parameter $(\gamma_0, a, [b_0])$, we have produced a gg acceptable $\mathbf{b}$-admissible LR pair (ϕ_0, γ_0) whose Kottwitz parameter has the form $(\gamma_0, a', [b_0'])$. Next we need to show that we can twist to get the correct components a and $[b_0]$. This is where the triviality of the Kottwitz parameter comes into play, essentially providing a global compatibility that allows us to find the twist.

We begin by establishing the fact that for a gg acceptable LR pair (ϕ, ε), we have a canonical inner twist between $I_{\phi,\varepsilon}$ and G_ε.

Lemma 4.4.6. *Let (ϕ, ε) be an acceptable LR pair, and suppose that ϕ^Δ is defined over $\mathbb{Q}$ and $\varepsilon \in G(\mathbb{Q})$ (in particular, this applies to any acceptable gg pair). Then the inclusion $I_{\phi,\varepsilon} \hookrightarrow G_\varepsilon$ over $\overline{\mathbb{Q}}$ is an isomorphism.*

Proof. Recall from 2.6.2 that $I_{\phi,\overline{\mathbb{Q}}}$ is the centralizer of (the image of) ϕ^Δ in $G_{\overline{\mathbb{Q}}}$, so our goal is to show that any element commuting with ε must commute with ϕ^Δ.

We begin by showing that any element commuting with ε must commute with $\phi^\Delta \circ \nu(p)$ and $\phi^\Delta \circ \nu(\infty)$.

At p: let $g \in G(\overline{\mathbb{Q}}_p)$ such that $\theta = \mathrm{Int}(g) \circ \phi(p) \circ \zeta_p$ is unramified, and let $b_\theta \in G(\mathbb{Q}_p^{\mathrm{ur}})$ defined by $\theta(d_\sigma) = b_\theta \rtimes \sigma$. Write $\varepsilon' = \mathrm{Int}(g)\varepsilon$. By Lemma 2.6.7 we have

$$\mathrm{Int}(g) \circ \phi^\Delta \circ \nu(p) = (\mathrm{Int}(g) \circ \phi(p) \circ \zeta_p)^\Delta = -\nu_{b_\theta},$$

so the centralizer of $\mathrm{Int}(g) \circ \phi^\Delta \circ \nu(p)$ is M_{b_θ}. On the other hand, (b_θ, ε') is a p-adic realization of our acceptable pair (ϕ, ε)—in particular, we can apply Lemma 2.2.5 to conclude that $G_{\varepsilon'} \subset M_{b_\theta}$.

Since the centralizer of $\mathrm{Int}(g) \circ \phi^\Delta \circ \nu(p)$ is M_{b_θ} we see the centralizer of $\phi^\Delta \circ \nu(p)$ is $\mathrm{Int}(g^{-1})M_{b_\theta}$, and likewise we have $G_\varepsilon = \mathrm{Int}(g^{-1})G_{\varepsilon'}$. The above analysis then shows that $\mathrm{Int}(g^{-1})G_{\varepsilon'} = \mathrm{Int}(g^{-1})M_{b_\theta}$, which is to say that the centralizer of ε is contained in the centralizer of $\phi^\Delta \circ \nu(p)$, as desired.

At ∞: as in the proof of [KSZ21, Lemma 4.1.10], the fact is that $\phi^\Delta \circ \nu(\infty)$ is central in G, and therefore any element commuting with ε trivially commutes with $\phi^\Delta \circ \nu(\infty)$.

Now, suppose that $g \in G(\overline{\mathbb{Q}})$ commutes with ε, and we want to see that g commutes with ϕ^Δ.

Recall that ϕ^Δ is a morphism $Q \to G$ where $Q = \varprojlim_L Q^L$ is the kernel of $\mathfrak{Q}$. For each finite Galois $L/\mathbb{Q}$, the torus Q^L is generated by the $\mathrm{Gal}(L/\mathbb{Q})$-conjugates of the images of $\nu(p)^L$ and $\nu(\infty)^L$. Thus Q is generated by the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of the images of $\nu(p)$ and $\nu(\infty)$.

For any $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the conjugate $\rho(g)$ again commutes with ε by our hypothesis that ε is rational, and therefore by the above arguments $\rho(g)$ commutes with $\phi^\Delta \circ \nu(v)$ for $v = p, \infty$. Applying ρ^{-1} and using our hypothesis that ϕ^Δ is defined over $\mathbb{Q}$, we see that g commutes with $\phi^\Delta \circ \rho^{-1}(\nu(v))$ for $v = p, \infty$. Since ρ was arbitrary and the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of the images of $\nu(v)$ generate Q, this implies that g commutes with ϕ^Δ, as desired. □

4.4.7 Let (ϕ, ε) be a gg acceptable pair, so that letting $q_\rho \in \mathfrak{Q}$ a lift of $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $\phi(q_\rho) = g_\rho \rtimes \rho$ with $g_\rho \in G_\varepsilon^\circ$. We know from 2.6.2 that over $\overline{\mathbb{Q}}$ the group I_ϕ is identified with $Z_G(\phi^\Delta)$, so we can rewrite the R-points for R a $\mathbb{Q}$-algebra as

$$I_\phi(R) = \{g \in Z_G(\phi^\Delta)(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \mid \mathrm{Int}(g) \circ \phi = \phi\}.$$

Writing out the condition $\mathrm{Int}(g) \circ \phi = \phi$ and evaluating at q_ρ, we get

$$g g_\rho \rho(g)^{-1} \rtimes \rho = g_\rho \rtimes \rho,$$

and rearranging this amounts to $g = g_\rho \rho(g) g_\rho^{-1}$. In other words, we are taking fixed points under the Galois action $g_\rho \rho(\cdot) g_\rho^{-1}$. (The action does not depend on the choice of q_ρ, as changing this choice changes g_ρ by an element of $\mathrm{im}(\phi^\Delta)$, and we are working in the centralizer of this image).

This shows that the group I_ϕ is defined as an inner form of $Z_G(\phi^\Delta)$ by the cocycle

$$\rho \mapsto \mathrm{Int}(g_\rho) \in \mathrm{Aut}((G_\varepsilon^\circ)_{\overline{\mathbb{Q}}}) \qquad \rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$$

In view of the isomorphism of Lemma 4.4.6, this produces—for any gg acceptable LR pair (ϕ, ε)—compatible inner twists

$$(I_{\phi,\varepsilon}^\circ)_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G_\varepsilon^\circ)_{\overline{\mathbb{Q}}},$$
$$(I_{\phi,\varepsilon})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} (G_\varepsilon)_{\overline{\mathbb{Q}}}$$

defined by the same cocycle.

4.4.8 On the Kottwitz parameter side, we recall a preparatory lemma in the new language of Kottwitz parameters.

Lemma 4.4.9. *Let $(\gamma_0, a, [b_0])$ and $(\gamma_0, a', [b_1])$ be acceptable* $\mathbf{b}$*-admissible Kottwitz parameters with the same element γ_0. Then $\nu_{b_0} = \nu_{b_1}$.*

Proof. Both Kottwitz parameters are assumed to be $\mathbf{b}$-admissible, so $[b_0]$ and $[b_1]$ both map to $\mathbf{b}$ in $B(G)$. In particular, b_0 and b_1 are σ-conjugate in $G(L)$. Furthermore, the semi-simple element γ_0 as an element of $G(L)$ lies in both $J_{b_0}(\mathbb{Q}_p)$ and $J_{b_1}(\mathbb{Q}_p)$, and is acceptable with respect to both. Thus we are in the situation of Lemma 2.2.6, and we conclude that $\nu_{b_0} = \nu_{b_1}$. □

Now we are prepared to show that every acceptable $\mathbf{b}$-admissible Kottwitz parameter with trivial invariant comes from an LR pair—without worrying about τ-twists for the moment.

Proposition 4.4.10. *Let $(\gamma_0, a, [b_0])$ be an acceptable* $\mathbf{b}$*-admissible Kottwitz parameter with trivial Kottwitz invariant, and suppose there is a gg acceptable* $\mathbf{b}$*-admissible pair (ϕ_0, γ_0). Then there is a gg acceptable* $\mathbf{b}$*-admissible pair (ϕ_1, ε_1) with*

$$\mathbf{t}(\phi_1, \varepsilon_1, 1) = (\gamma_0, a, [b_0]).$$

This is Theorem 4.8.9 of [KSZ21], except that their "p^n-admissible" hypothesis has been replaced by our " acceptable $\mathbf{b}$-admissible" hypothesis. We briefly sketch how their proof carries over to our case.

Proof. Write $\mathbf{t}(\phi_0, \gamma_0, 1) = (\gamma_0, a', [b'_0])$, and $I_0 = G^\circ_{\gamma_0}$. By Lemma 4.3.16, our hypothesis that (ϕ_0, γ_0) is acceptable and $\mathbf{b}$-admissible implies that $(\gamma_0, a', [b'_0])$ is acceptable and $\mathbf{b}$-admissible.

Since γ_0 is acceptable with respect to $[b'_0]$, by Lemma 2.2.5 we have $I_0 \subset M_{b'_0}$, and therefore $\nu_{b'_0}$ is central in I_0. This is the first ingredient; we also need the fact from Lemma 4.4.9 that $\nu_{b_0} = \nu_{b'_0}$; and the fact from 4.4.7 of a canonical inner twisting $(I^\circ_{\phi_0,\gamma_0})_{\overline{\mathbb{Q}}} \xrightarrow{\sim} I_{0,\overline{\mathbb{Q}}}$.

In the presence of these three ingredients, the proof of [KSZ21, Thm 4.8.9] carries over without modification to show the existence of a gg semi-admissible LR pair (ϕ_1, ε_1) with $\mathbf{t}(\phi_1, \varepsilon_1, 1) = (\gamma_0, a, [b_0])$. By Lemma 4.3.16, (ϕ_1, ε_1) is acceptable and $\mathbf{b}$-admissible. □

The last step is to incorporate τ-twists. We collect the full result in the following proposition.

Proposition 4.4.11. *Let $(\gamma_0, a, [b_0])$ be an acceptable* $\mathbf{b}$*-admissible Kottwitz parameter with trivial Kottwitz invariant, and $\tau \in \Gamma(\mathfrak{E}^p)_0$ a tori-rational element. Then there is a gg acceptable* $\mathbf{b}$*-admissible LR pair (ϕ, ε) such that*

$$\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0]).$$

This is our analogue of [KSZ21, Prop 5.2.7], and the proof there carries over to our case with the appropriate changes, but for convenience we perform the changes.

Proof. By Proposition 4.4.4 there exists a gg acceptable **b**-admissible pair (ϕ_0, γ_0), and furthermore by Proposition 4.4.10 there is a gg acceptable **b**-admissible pair (ϕ_1, ε_1) with $\mathbf{t}(\phi_1, \varepsilon_1, 1) = (\gamma_0, a, [b_0])$.

Write

$$(\gamma_0, a', [b_0']) = \mathbf{t}(\phi_1, \varepsilon_1, \tau(\phi_1)).$$

By [KSZ21, Prop 4.8.2] we have equality of p-parts $[b_0'] = [b_0]$, and the difference between the away-from-p parts $a' - a$ is equal to the image of $\tau(\phi_1)$ in $\mathfrak{E}(I_0, G; \mathbb{A}_f^p)$.

Choose a maximal torus $T \subset I_{\phi_1}$ containing γ_0. Our hypothesis that τ is torirational implies that we can choose $\beta \in Ш_G^{\infty,p}(\mathbb{Q}, T)$ so that $\tau(\phi_1)$ and β have the same image in $H^1(\mathbb{A}_f^p, T)$.

The map $T \hookrightarrow I^\circ_{\phi_1,\varepsilon_1}$ induces a map $Ш_G^{\infty,p}(\mathbb{Q}, T) \to Ш_G^{\infty,p}(\mathbb{Q}, I^\circ_{\phi_1,\varepsilon_1})$. Let $e \in Z^1(\mathbb{Q}, I^\circ_{\phi_1,\varepsilon_1})$ be a cocycle representing the image of $-\beta$ in $Ш_G^{\infty,p}(\mathbb{Q}, I^\circ_{\phi_1,\varepsilon_1})$. By [KSZ21, Prop 4.3.7] the twist $e\phi_1$ is again gg and admissible. Define $(\phi, \varepsilon) = (e\phi_1, \varepsilon_1)$.

Write

$$(\gamma_0, a'', [b_0'']) = \mathbf{t}(\phi, \varepsilon, \tau(\phi)).$$

Compared to $\mathbf{t}(\phi_1, \varepsilon_1, \tau(\phi_1))$, the p-parts are equal $[b_0''] = [b_0']$ because $-\beta$ is trivial at p, and the difference in the away-from-p parts $a'' - a'$ is equal to the image of $-\beta$ in $\mathfrak{E}(I_0, G; \mathbb{A}_f^p)$ by [KSZ21, Prop 5.2.4] (note that the relevant part of that proposition does not rely on their "p^n-admissible" hypothesis which is missing from our hypotheses). This shows that $a'' = a$ and $[b_0''] = [b_0]$, so that $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ as desired. The pair (ϕ, ε) is then also acceptable and **b**-admissible by Lemma 4.3.16. □

4.4.12 Having completed this result, and combining it with the discussion at the beginning of this section, we conclude that under the map (for $\tau \in \Gamma(\mathfrak{E}^p)_0$ torirational)

$$\mathbf{t} : \mathcal{LRP}_{\mathrm{sa}}/\mathrm{conj.} \to \mathcal{KP}/\mathrm{isom.}$$

the image of the set of acceptable **b**-admissible LR pairs is the set of acceptable **b**-admissible Kottwitz parameters with trivial Kottwitz invariant.

4.5 Point-Counting Formula

In this section, after a few more technical requirements, we arrive at the final form of the unstable trace formula for Igusa varieties of Hodge type.

Recall the result of our preliminary point counting as summarized in Proposition 4.2.10, updated to the language of LR pairs. Let $\tau \in \Gamma(\mathcal{H})_0$ a tori rational element satisfying Theorem 3.6.2, which we may lift to a tori-rational element of $\Gamma(\mathfrak{E}^p)_0$ (still called τ by abuse of notation). For any acceptable function $f \in C_c^\infty(G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p))$ we have

$$\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \sum_{(\phi,\varepsilon)} \frac{\operatorname{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q}) \backslash G^\circ_{b,\gamma\times\delta}\right)}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})]} O^{G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p)}_{\gamma\times\delta}(f) \operatorname{tr}(\xi(\varepsilon)), \quad (4.5.1)$$

where (ϕ, ε) ranges over conjugacy classes of **b**-admissible LR pairs, and γ, δ are the elements appearing in the classical Kottwitz parameter associated to (ϕ, ε) and $\tau(\phi)$. In fact, as a first modification, we can restrict the sum to pairs (ϕ, ε) which are furthermore acceptable: since an acceptable function f is supported on acceptable elements of $J_b(\mathbb{Q}_p)$, the orbital integral will be zero unless (ϕ, ε) is acceptable.

We begin with a technical lemma which will help us work with our automorphism groups at p.

Lemma 4.5.2. *Let $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ be a morphism of Galois gerbs, and suppose that ϕ^Δ factors through the center of G. Then $\phi(p) \circ \zeta_p$ is conjugate to an unramified morphism θ, and writing $\theta^{\mathrm{ur}}(d_\sigma) = b_0 \rtimes \sigma$, we have $I_\phi \cong J_{b_0}$ over $\mathbb{Q}_p$.*

Proof. As in 2.6.6, any morphism $\mathfrak{G}_p \to \mathfrak{G}_G(p)$ is conjugate to an unramified morphism, and this applies to $\phi(p) \circ \zeta_p$. For θ such an unramified morphism, Lemma 2.6.7 states that $I_\theta \cong J_{b_0}$ over $\mathbb{Q}_p$. Furthermore if $u \in G(\overline{\mathbb{Q}}_p)$ verifies $\operatorname{Int}(u) \circ \phi(p) \circ \zeta_p = \theta$, then $x \mapsto \operatorname{Int}(u)x$ gives a $\mathbb{Q}_p$-isomorphism $I_{\phi(p)\circ\zeta_p} \xrightarrow{\sim} I_\theta$. Thus we have

$$I_{\phi(p)\circ\zeta_p} \xrightarrow{\sim} I_\theta \xrightarrow{\sim} J_{b_0},$$

it remains for us to show that $I_\phi \cong I_{\phi(p)\circ\zeta_p}$ over $\mathbb{Q}_p$.

Recall as in 4.4.7 that I_ϕ is an inner form of $Z_G(\phi^\Delta)$ defined by the cocycle $\rho \mapsto \operatorname{Int}(g_\rho)$. As we assume ϕ^Δ factors through the center of G, we have $Z_G(\phi^\Delta) = G$. Similarly, $I_{\phi(p)\circ\zeta_p}$ is an inner form of $Z_{G_{\mathbb{Q}_p}}((\phi(p) \circ \zeta_p)^\Delta) = G_{\mathbb{Q}_p}$ defined by the same cocycle $\rho \mapsto \operatorname{Int}(g_\rho)$. Thus $I_\phi \cong I_{\phi(p)\circ\zeta_p}$ over $\mathbb{Q}_p$ as desired. □

4.5.3 In order to translate the volume term in 4.5.1, we need to define a group associated to a Kottwitz parameter which will play the role of $I^\circ_{\phi,\varepsilon}$.

Let $\mathfrak{c} = (\gamma_0, a, [b_0])$ be a Kottwitz parameter with $[b_0] \in B(I_0)$ basic, and with trivial Kottwitz parameter. We can define an inner form $I_\mathfrak{c}$ of $I_0 = G^\circ_{\gamma_0}$ as follows. Writing $a = (a_\ell)_{\ell \neq p,\infty}$, let I_ℓ be the inner form of I_0 over $\mathbb{Q}_\ell$ defined by a_ℓ (or to be precise, the image of a_ℓ in $H^1(\mathbb{Q}_\ell, I_0^{\mathrm{ad}})$). At p, let $I_p = J_{b_0}^{I_0}$ be the inner form of I_0 over $\mathbb{Q}_p$ defined by the basic class $[b_0] \in B(I_0)$. At ∞, let I_∞ be the inner form of I_0 over $\mathbb{R}$ which is compact modulo Z_G (concretely, this form is defined by the Cartan involution $\mathrm{Int}(h(i))$ on I_0 where $T \subset I_{0,\mathbb{R}}$ and $h \in X_*(T)$ are as in the construction of the Kottwitz invariant 4.3.11). By [KSZ21, Prop 4.7.8], these local components determine a unique inner form I_c of I_0 over $\mathbb{Q}$ such that $I_{\mathfrak{c}} \otimes \mathbb{Q}_v \cong I_v$ for all places v of $\mathbb{Q}$.

Lemma 4.5.4. *Let (ϕ, ε) a gg acceptable $\mathbf{b}$-admissible LR pair, $\tau(\phi) \in I_\phi(\overline{\mathbb{A}}_f^p)$, and $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ the associated Kottwitz parameter.*

1. *Then $[b_0]$ is basic in $B(I_0)$ and $\mathfrak{c} = (\gamma_0, a, [b_0])$ has trivial Kottwitz parameter, so we can define an inner form $I_{\mathfrak{c}}$ of $I_0 = G_{\gamma_0}^\circ (= G_\varepsilon^\circ)$ as above.*

2. *There is a $\mathbb{Q}$-isomorphism $I_{\mathfrak{c}} \cong I_{\phi,\varepsilon}^\circ$.*

Proof. We have assumed (ϕ, ε) is acceptable, thus $(\gamma_0, a, [b_0])$ is as well. Applying Lemma 2.2.5 to γ_0 and b_0 we have $I_0 \subset M_{b_0}$, which implies that I_0 centralizes ν_{b_0}, i.e. $[b_0]$ is basic in $B(I_0)$. Furthermore by Lemma 4.4.1, the Kottwitz invariant of $(\gamma_0, a, [b_0])$ is trivial. This verifies item 1.

Since $I_{\mathfrak{c}}$ is characterized uniquely by its local components, in order to prove item 2 it suffices to show that $I_{\phi,\varepsilon}^\circ$ is isomorphic to I_v over $\mathbb{Q}_v$ for all places v of $\mathbb{Q}$ (where I_v is defined as in 4.5.3). We know that I_c and $I_{\phi,\varepsilon}^\circ$ are inner forms of $I_0 = G_\varepsilon^\circ$ (the former by definition, and the latter by 4.4.7), so we will identify the local components by comparing the cocycles defining each as an inner form.

At $\ell \neq p, \infty$: recall from 2.6.12 that the cocycle $\zeta_\phi^{p,\infty}$ defines for each $\ell \neq p, \infty$ a cocycle

$$\zeta_{\phi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\zeta_\phi^{p,\infty}} G(\overline{\mathbb{A}}_f^p) \to G(\overline{\mathbb{Q}}_\ell),$$

and we have $(\phi(\ell) \circ \zeta_\ell)(\rho) = \zeta_{\phi,\ell}(\rho) \rtimes \rho$ for all $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. The cocycle $a = (a_\ell)_\ell$ defining $I_{\mathfrak{c}}$ at the places ℓ was defined in 4.3.13 to be (the image of) the cocycle

$$A(\rho) = \tau_G^{-1} \zeta_\phi^{p,\infty}(\rho) \rho(\tau_G)$$

where τ_G is the image of $\tau(\phi)$ in G.

Recall from 4.4.7 that, letting $q_\rho \in \mathfrak{Q}$ a lift of $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and writing $\phi(q_\rho) = g_\rho \rtimes \rho$, we have $I_{\phi,\varepsilon}^\circ$ as an inner form of G_ε° defined by the cocycle $\rho \mapsto \mathrm{Int}(g_\rho)$. Furthermore, we can choose $q_\rho = \zeta_\ell(\rho)$ in order to have $(\phi(\ell) \circ \zeta_\ell)(\rho) = g_\rho \rtimes \rho$ for $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. That is, $g_\rho = \zeta_{\phi,\ell}(\rho)$. This shows that $\mathrm{Int}(\tau_G)$ gives an isomorphism between $I_{\phi,\varepsilon}^\circ$ and I_ℓ over $\mathbb{Q}_\ell$.

At p: as (ϕ, ε) is gg, we can factor ϕ as

$$\mathfrak{Q} \xrightarrow{\phi_0} \mathfrak{G}_{G_\varepsilon^\circ} \longrightarrow \mathfrak{G}_G.$$

Then I_{ϕ_0}, defined on a $\mathbb{Q}$-algebra R by

$$I_{\phi_0}(R) = \{g \in G_\varepsilon^\circ(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \mid \mathrm{Int}(g) \circ \phi_0 = \phi_0\},$$

is an inner form of $G_\varepsilon^\circ = Z_{G_\varepsilon^\circ}(\phi_0^\Delta)$ (note since (ϕ, ε) is gg and acceptable, Lemma 4.4.6 applies to show G_ε° centralizes ϕ^Δ) defined by the same cocycle as $I_{\phi,\varepsilon}^\circ$, which is to say that $I_{\phi_0} \cong I_{\phi,\varepsilon}^\circ$ over $\mathbb{Q}$. On the other hand, ϕ_0 satisfies the hypotheses of Lemma 4.5.2 (with our (I_0, b_0) in place of the (G, b_0) there, and recalling that $\mathrm{im}(\phi_0^\Delta) = \mathrm{im}(\phi^\Delta)$ is central in I_0), so we conclude that $I_{\phi_0} \cong J_{b_0}^{I_0}$ over $\mathbb{Q}_p$. Thus $I_{\phi,\varepsilon}^\circ \cong I_p = J_{b_0}^{I_0}$ over $\mathbb{Q}_p$.

At ∞: as ϕ is admissible, [KSZ21, Lemma 3.1.9] tells us that $(I_\phi/Z_G)(\mathbb{R})$ is compact, and we also know that $(I_\infty/Z_G)(\mathbb{R})$ is compact. Thus by $I_{\phi,\varepsilon}^\circ$ and I_∞ must be the same inner form of I_0 over $\mathbb{R}$.

Having shown that $I_\mathfrak{c}$ and $I_{\phi,\varepsilon}^\circ$ are isomorphic over $\mathbb{Q}_v$ for all places v of $\mathbb{Q}$, we conclude that they are in fact isomorphic over $\mathbb{Q}$. □

Now we can rewrite the volume term in 4.5.1.

Lemma 4.5.5. *Let (ϕ, ε) be a gg acceptable* **b***-admissible LR pair, $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, and $\mathfrak{c} = \mathbf{t}(\phi, \varepsilon, \tau(\phi))$ the associated Kottwitz parameter. Write $G_b = G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$, and $\gamma \times \delta$ for the image of* $\mathrm{Int}(\tau(\phi))\varepsilon$ *in G_b. Then*

$$\mathrm{vol}(I_{\phi,\varepsilon}^\circ(\mathbb{Q}) \backslash G_{b,\gamma\times\delta}^\circ) = \mathrm{vol}(I_\mathfrak{c}(\mathbb{Q}) \backslash I_\mathfrak{c}(\mathbb{A}_f)).$$

Proof. We know from 4.4.6 that the map $I_{\phi,\varepsilon} \to G_\varepsilon$ is an isomorphism. Thus we can regard G_ε as lying in (the image of) I_ϕ, and so $\mathrm{Int}(\tau(\phi))$ gives an $\mathbb{A}_f^p$-isomorphism $(I_{\phi,\varepsilon} \cong) G_\varepsilon \cong G_\gamma$ (of course taking $I_{\phi,\varepsilon}^\circ(\mathbb{Q})$ to its $\tau(\phi)$-twist). This shows that

$$\mathrm{vol}(I_{\phi,\varepsilon}^\circ(\mathbb{Q}) \backslash G_{b,\gamma\times\delta}^\circ) = \mathrm{vol}(I_{\phi,\varepsilon}^\circ(\mathbb{Q}) \backslash I_{\phi,\varepsilon}^\circ(\mathbb{A}_f))$$

(note that $\tau(\phi)$ is trivial at p, so we need only worry about the components away from p).

Now Lemma 4.5.4 gives us

$$\mathrm{vol}(I_{\phi,\varepsilon}^\circ(\mathbb{Q}) \backslash I_{\phi,\varepsilon}^\circ(\mathbb{A}_f)) = \mathrm{vol}(I_\mathfrak{c}(\mathbb{Q}) \backslash I_\mathfrak{c}(\mathbb{A}_f)),$$

and combining this with the above equality, the result follows. □

4.5.6 Next we examine the orbital integral terms. Recall that in the setting of our preliminary formula 4.5.1 we have defined $\gamma \times \delta \in G_b = G(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p)$ to be the image of $\mathrm{Int}(\tau(\phi))\varepsilon$; or in the language of 4.3.17, γ, δ come from the classical Kottwitz parameter $(\gamma_0, \gamma, \delta)$ associated to (ϕ, ε) and $\tau(\phi)$. If $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) =$

$(\gamma_0, a, [b_0])$ is the Kottwitz parameter associated to our LR pair, then as in 4.3.18 we can construct a classical Kottwitz parameter from $(\gamma_0, a, [b_0])$, and by Lemma 4.3.19 this classical Kottwitz parameter produces the same elements γ, δ (up conjugacy in $G(\mathbb{A}_f^p)$ or $J_b(\mathbb{Q}_p)$ respectively). Thus, orbital integrals defined in terms of classical Kottwitz parameters on both sides will agree.

Define

$$T_\xi^f(\phi, \varepsilon, \tau(\phi)) = \mathrm{vol}\left(I^\circ_{\phi,\varepsilon}(\mathbb{Q})\backslash G^\circ_{b,\gamma\times\delta}\right) O_{\gamma\times\delta}^{G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p)}(f)\,\mathrm{tr}(\xi(\varepsilon))$$

where γ, δ are the elements in the classical Kottwitz parameter associated to (ϕ, ε) and $\tau(\phi)$; and

$$T_\xi^f(\gamma_0, a, [b_0]) = \mathrm{vol}\left(I^\circ_{\mathfrak{c}}(\mathbb{Q})\backslash I^\circ_{\mathfrak{c}}(\mathbb{A}_f)\right) O_{\gamma\times\delta}^{G(\mathbb{A}_f^p)\times J_b(\mathbb{Q}_p)}(f)\,\mathrm{tr}(\xi(\gamma_0))$$

where γ, δ are the elements in the classical Kottwitz parameter associated to $(\gamma_0, a, [b_0])$, and $I_{\mathfrak{c}}$ is the group associated to $\mathfrak{c} = (\gamma_0, a, [b_0])$ as in 4.5.3.

Lemma 4.5.7. *Let (ϕ, ε) be a gg acceptable* **b***-admissible LR pair, $\tau(\phi) \in I_\phi^{\mathrm{ad}}(\mathbb{A}_f^p)$, and* $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) = (\gamma_0, a, [b_0])$ *the associated Kottwitz parameter. Then*

$$T_\xi^f(\phi, \varepsilon, \tau(\phi)) = T_\xi^f(\gamma_0, a, [b_0]).$$

Proof. The volume terms are equal by Lemma 4.5.5; the orbital integrals are equal by 4.3.19 and the discussion in 4.5.6; and $\mathrm{tr}(\xi(\varepsilon)) = \mathrm{tr}(\xi(\gamma_0))$ because, by the construction in 4.3.13, we have $\varepsilon = \gamma_0$. □

4.5.8 We now return to our study of the map $\mathbf{t} : \mathcal{LRP}/\mathrm{conj.} \to \mathcal{KP}/\mathrm{isom}$. We showed in §4.4 that the image of the set of acceptable **b**-admissible LR pairs is the set of acceptable **b**-admissible Kottwitz parameters with trivial Kottwitz invariant. Our next goal is to examine the fibers of this map; we will understand them in terms of cohomological twists.

Recall from 2.6.2 that we can twist a morphism $\phi : \mathfrak{Q} \to \mathfrak{G}_G$ by a cocycle $e \in Z^1(\mathbb{Q}, I_\phi)$ to get another morphism $e\phi$. We saw in 2.6.11 that if ϕ is admissible, then $e\phi$ is again admissible exactly when e lies in $\text{Ш}_G^{p,\infty}(\mathbb{Q}, I_\phi)$.

We now consider twisting LR pairs. For (ϕ, ε) an LR pair and $e \in Z^1(\mathbb{Q}, I_{\phi,\varepsilon}) \subset Z^1(\mathbb{Q}, I_\phi)$, we can define a twist by simply twisting the morphism $(e\phi, \varepsilon)$. By [KSZ21, Lemma 4.2.5] this is again an LR pair—essentially we need to take a cocycle in $I_{\phi,\varepsilon}$ rather than I_ϕ to ensure that ε still lies in I_ϕ. As in the case of twisting morphisms, two twists $(e\phi, \varepsilon)$ and $(e'\phi, \varepsilon)$ are conjugate by $G(\overline{\mathbb{Q}})$ exactly when e, e' define the same class in $H^1(\mathbb{Q}, I_{\phi,\varepsilon})$ ([KSZ21, Lemma 4.2.6]).

Now, suppose that (ϕ, ε) is gg. Then [KSZ21, Lemma 4.2.5] also tells us that if $e \in Z^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon}) \subset Z^1(\mathbb{Q}, I_{\phi,\varepsilon})$ then the twist $(e\phi, \varepsilon)$ is again gg. This gives us, for any gg pair (ϕ, ε), a map

$$Z^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon}) \to \mathcal{LRP}^{\mathrm{gg}}.$$

If we only consider LR pairs up to conjugacy then this map factors through $H^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon})$, giving

$$H^1(\mathbb{Q}, I^\circ_{\phi,\varepsilon}) \to \mathcal{LRP}^{\mathrm{gg}}/\mathrm{conj}.$$

The point of all this is that each fiber of the map $\mathbf{t}$ is contained in the image of one such map—that is, two elements in the same fiber are always related by an H^1-twist. In order to show this, we need some preparatory lemmas.

Lemma 4.5.9. *Let (ϕ, ε) a gg acceptable LR pair, and $\mathrm{Int}(g)\varepsilon \in G(\mathbb{Q})$ a rational element stably conjugate to ε, i.e. $g \in G(\overline{\mathbb{Q}})$ and $g^{-1}\rho(g) \in G^\circ_\varepsilon$ for $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the conjugate $(\mathrm{Int}(g) \circ \phi, \mathrm{Int}(g)\varepsilon)$ is again gg and acceptable.*

Proof. Acceptability is insensitive to conjugacy, so $(\mathrm{Int}(g) \circ \phi, \mathrm{Int}(g)\varepsilon)$ is acceptable. (We include this condition in order to apply Lemma 4.4.6 below).

To see that it is gg, we verify the three conditions of Definition 4.3.5 in turn.

To show that $(\mathrm{Int}(g) \circ \phi)^\Delta = \mathrm{Int}(g) \circ \phi^\Delta$ is defined over $\mathbb{Q}$, we want to check

$$\rho(\mathrm{Int}(g) \circ \phi^\Delta) = \mathrm{Int}(g) \circ \phi^\Delta$$

for $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since (ϕ, ε) is gg, we have ϕ^Δ defined over $\mathbb{Q}$. Thus we can rewrite the left hand side of the above equation as $\mathrm{Int}(\rho(g)) \circ \phi^\Delta$, and rearranging the above equation becomes

$$\mathrm{Int}(g^{-1}\rho(g)) \circ \phi^\Delta = \phi^\Delta.$$

By our stable conjugacy assumption we have $g^{-1}\rho(g) \in G^\circ_\varepsilon$, and Lemma 4.4.6 this implies that $g^{-1}\rho(g)$ commutes with ϕ^Δ. This verifies the first condition.

We have assumed that $\mathrm{Int}(g)\varepsilon$ is defined over $\mathbb{Q}$, and it is semi-simple and elliptic in $G(\mathbb{R})$ because it is conjugate to the element ε which satisfies those conditions. This verifies the second condition.

Finally we want to show that, writing $\mathrm{Int}(g) \circ \phi(q_\rho) = x_\rho \rtimes \rho$, we have $x_\rho \in G^\circ_{\mathrm{Int}(g)\varepsilon}$ (where $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $q_\rho \in \mathfrak{Q}$ is a lift of ρ). Write $\phi(q_\rho) = y_\rho \rtimes \rho$. Then we have

$$x_\rho = gy_\rho\rho(g)^{-1} = gy_\rho g^{-1} \cdot g\rho(g)^{-1}.$$

In the right-most form we observe that $gy_\rho g^{-1} \in G^\circ_{\mathrm{Int}(g)\varepsilon}$ since $y_\rho \in G^\circ_\varepsilon$, and $g\rho(g)^{-1} \in G^\circ_{\mathrm{Int}(g)\varepsilon}$ by our stable conjugacy assumption. This shows $x_\rho \in G^\circ_{\mathrm{Int}(g)\varepsilon}$, as desired. □

Lemma 4.5.10. *Let (ϕ, ε) and (ϕ', ε) be gg acceptable* **b**-*admissible LR pairs with the same element $\varepsilon \in G(\mathbb{Q})$. Then $\phi^\Delta = \phi'^\Delta$.*

Proof. By the same reasoning as the last paragraphs of the proof of Lemma 4.4.6, it suffices to show that $\phi^\Delta \circ \nu(v) = \phi'^\Delta \circ \nu(v)$ for $v = p, \infty$ (this implies that ϕ^Δ and ϕ'^Δ agree on a generating set for $\mathfrak{Q}^\Delta = Q$, and being morphisms they must then agree everywhere).

At ∞: by [KSZ21, Lemma 4.1.10], simply the fact that both ϕ and ϕ' are admissible implies that $\phi^\Delta \circ \nu(\infty) = \phi'^\Delta \circ \nu(\infty)$.

At p: since (ϕ, ε) is gg, we can factor

$$\phi : \mathfrak{Q} \to \mathfrak{G}_{G_\varepsilon^\circ} \to \mathfrak{G}_G,$$

and therefore we can conjugate $\phi(p) \circ \zeta_p$ to an unramified morphism θ by an element $u \in G_\varepsilon^\circ(\overline{\mathbb{Q}})$. Let $b_\theta \in G_\varepsilon^\circ(\mathbb{Q}_p^{\mathrm{ur}})$ be the corresponding element as in 2.6.5. In the same way we can conjugate $\phi'(p) \circ \zeta_p$ by an element $u' \in G_\varepsilon^\circ(\overline{\mathbb{Q}})$ and produce an element $b'_\theta \in G_\varepsilon^\circ(\mathbb{Q}_p^{\mathrm{ur}})$. Then (b_θ, ε) is a p-adic realization of (ϕ, ε), and (b'_θ, ε) is a p-adic realization of (ϕ', ε).

We have assumed that our LR pairs are acceptable and **b**-admissible, which implies that b_θ, b'_θ and ε satisfy the hypotheses of Lemma 2.2.6, and we conclude that $\nu_{b_\theta} = \nu_{b'_\theta}$.

On the other hand, we have

$$\mathrm{Int}(u) \circ \phi^\Delta \circ \nu(p) = (\mathrm{Int}(u) \circ \phi(p) \circ \zeta_p)^\Delta \stackrel{*}{=} -\nu_{b_\theta} = -\nu_{b'_\theta} \stackrel{*}{=} (\mathrm{Int}(u') \circ \phi'(p) \circ \zeta_p)^\Delta = \mathrm{Int}(u') \circ \phi'^\Delta \circ \nu($$

where the starred equalities are given by Lemma 2.6.7. By our acceptable hypothesis, we can apply Lemma 2.2.5 to see that G_ε commutes with $\nu_{b_\theta} = \nu_{b'_\theta}$. But the above equation demonstrates that $\phi^\Delta \circ \nu(p)$ and $-\nu_{b_\theta}$ and $-\nu_{b'_\theta}$ and $\phi'^\Delta \circ \nu(p)$ are all conjugate by $G_\varepsilon^\circ(\overline{\mathbb{Q}})$, so they must all be equal, and in particular $\phi^\Delta \circ \nu(p) = \phi'^\Delta \circ \nu(p)$. □

Now we are prepared to show that points in the same fiber of **t** are related by an H^1-twist.

Lemma 4.5.11. *Suppose that (ϕ, ε) and (ϕ', ε') are gg acceptable* **b**-*admissible LR pairs which give rise to isomorphic Kottwitz parameters. Then the conjugacy classes of (ϕ, ε) and (ϕ', ε') are related by twisting by an element of $H^1(\mathbb{Q}, I_{\phi,\varepsilon}^\circ)$.*

Proof. In fact we only need to assume that the rational elements, say γ_0 and γ_0', appearing in the two Kottwitz parameters are stably conjugate (which follows from the isomorphism of Kottwitz parameters)—this justifies our neglect of τ-twists in the statement, as τ-twists do not affect the rational element γ_0 of the Kottwitz parameter.

So our hypothesis implies that ε and ε' are stably conjugate, and by Lemma 4.5.9 we can conjugate (ϕ', ε') to a gg pair (ϕ_0, ε) which is again acceptable and **b**-admissible.

By Lemma 4.5.10 we have $\phi^\Delta = \phi_0^\Delta$. As in Lemma 2.6.2 we can choose $e \in Z^1(\mathbb{Q}, I_\phi)$ so that $\phi_0 = e\phi$. Now write

$$\phi(q_\rho) = g_\rho \rtimes \rho, \qquad \phi_0(q_\rho) = e\phi(q_\rho) = e_\rho g_\rho \rtimes \rho$$

where as usual $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $q_\rho \in \mathfrak{Q}$ is a lift of ρ. Since our LR pairs are gg, we have $g_\rho \in G_\varepsilon^\circ$ and $e_\rho g_\rho \in G_\varepsilon^\circ$, so we conclude that $e_\rho \in G_\varepsilon^\circ$. By 4.4.6 this shows that in fact $e \in Z^1(\mathbb{Q}, I_{\phi,\varepsilon}^\circ)$, demonstrating that the conjugacy classes of (ϕ, ε) and (ϕ', ε') are related by twisting by $H^1(\mathbb{Q}, I_{\phi,\varepsilon}^\circ)$. □

We also have an analogue of Proposition 2.6.11 characterizing which twists of a semi-admissible LR pair are semi-admissible.

Proposition 4.5.12 ([KSZ21, Prop 4.3.7])**.** *If (ϕ, ε) is gg and semi-admissible and $e \in Z^1(\mathbb{Q}, I_{\phi,\varepsilon}^\circ)$, then the twist $(e\phi, \varepsilon)$ is gg and semi-admissible exactly when e lies in $Ш_G^\infty(\mathbb{Q}, I_{\phi,\varepsilon}^\circ)$.*

Combining Proposition 4.5.12 with the discussion of 4.5.8, we have for each gg acceptable **b**-admissible pair (ϕ, ε) a map

$$\eta_{\phi,\varepsilon} : Ш_G^\infty(\mathbb{Q}, I_{\phi,\varepsilon}^\circ) \to \mathcal{LRP}_{\mathrm{sa}}^{\mathrm{gg}}/\mathrm{conj}.$$

and by Lemma 4.5.11 any two such LR pairs giving rise to isomorphic Kottwitz parameters must both lie in the image of one such map $\eta_{\phi,\varepsilon}$.

4.5.13 Let $\tau \in \Gamma(\mathfrak{E}^p)_0$ tori-rational, and $(\gamma_0, a, [b_0])$ an acceptable **b**-admissible Kottwitz parameter with trivial Kottwitz invariant. By 4.4.11, there is a gg acceptable **b**-admissible LR pair (ϕ_0, γ_0) such that $\mathbf{t}(\phi_0, \gamma_0, \tau(\phi_0)) = (\gamma_0, a, [b_0])$. Furthermore combining Proposition 4.5.12 and Lemma 4.5.11, we see that every gg acceptable **b**-admissible LR pair (ϕ, ε) satisfying $\mathbf{t}(\phi, \varepsilon, \tau(\phi)) \cong (\gamma_0, a, [b_0])$ is contained in the image of the map η_{ϕ_0,γ_0}, i.e. is related to (ϕ_0, γ_0) by twisting by $Ш_G^\infty(\mathbb{Q}, I_{\phi_0,\gamma_0}^\circ)$. Let $D(\phi_0, \gamma_0) \subset Ш_G^\infty(\mathbb{Q}, I_{\phi_0,\gamma_0}^\circ)$ be the subset of classes e such that

$$\mathbf{t}(e\phi_0, \gamma_0, \tau(e\phi_0)) \cong \mathbf{t}(\phi_0, \gamma_0, \tau(\phi_0)) = (\gamma_0, a, [b_0]),$$

i.e. twists that do not change the Kottwitz parameter (up to isomorphism). Then the map

$$D(\phi_0, \gamma_0) \hookrightarrow Ш_G^\infty(\mathbb{Q}, I_{\phi_0,\gamma_0}^\circ) \xrightarrow{\eta_{\phi_0,\gamma_0}} \mathcal{LRP}_{\mathrm{sa}}^{\mathrm{gg}}/\mathrm{conj}.$$

is a surjection onto the set of conjugacy classes of LR pairs whose associated Kottwitz parameter is isomorphic to $(\gamma_0, a, [b_0])$. Note that it may not be a bijection, i.e. for each $e \in D(\phi_0, \gamma_0)$ we must account for the number of other elements of $D(\phi_0, \gamma_0)$ which give rise to the same conjugacy class of LR pairs, which number is $\#\{\text{fiber of } \eta_{\phi_0,\gamma_0} \text{ containing } e\}$. We proceed to count these sets, as they will appear en route to our final formula.

Lemma 4.5.14. *In the setting of 4.5.13, we have*

$$\#\{\textit{fiber of } \eta_{\phi_0,\gamma_0} \textit{ containing } e\} = \frac{|(I_{e\phi_0,\gamma_0}/I^{\circ}_{e\phi_0,\gamma_0})(\mathbb{Q})|}{[I_{e\phi_0,\gamma_0}(\mathbb{Q}) : I^{\circ}_{e\phi_0,\gamma_0}(\mathbb{Q})]}.$$

Note that $(e\phi_0, \gamma_0)$ is a gg acceptable pair (since its corresponding Kottwitz parameter is acceptable), so we are in the situation of 4.4.7 and we can equally well replace $I_{e\phi_0,\gamma_0}$ and $I^{\circ}_{e\phi_0,\gamma_0}$ by G_{γ_0} and $G^{\circ}_{\gamma_0}$, respectively.

Proof. This is proven in the last paragraph of the proof of Lemma 5.5.8 in [KSZ21] (in their notation, $\iota_H(\varepsilon) = [H_\varepsilon(\mathbb{Q}) : H^{\circ}_\varepsilon(\mathbb{Q})]$ and $\bar{\iota}_H(\varepsilon) = |(H_\varepsilon/H^{\circ}_\varepsilon)(\mathbb{Q})|$). □

Lemma 4.5.15. *In the setting of 4.5.13, we have*

$$|D(\phi_0, \gamma_0)| = \sum_{(a',[b'_0])} |Ш_G(\mathbb{Q}, I^{\circ}_{\phi_0,\gamma_0})|$$

where the sum runs over pairs $(a', [b'_0])$ such that $(\gamma_0, a', [b'_0])$ is an acceptable **b***-admissible Kottwitz parameter with trivial Kottwitz invariant, and $Ш_G$ is defined as in 2.6.10.*

As in Lemma 4.5.14, we can equally well replace $I^{\circ}_{\phi_0,\gamma_0}$ by $G^{\circ}_{\gamma_0}$.

Proof. We follow the proof of [KSZ21, Lemma 5.5.8], making changes as necessary for our situation.

Having fixed an LR pair (ϕ_0, γ_0) (and *not* simply a conjugacy class), [KSZ21, Cor 5.2.5] tells us that twisting by an element $e \in Ш^{\infty}_G(\mathbb{Q}, I^{\circ}_{\phi_0,\gamma_0})$ results in a well-defined Kottwitz parameter $\mathbf{t}(e\phi_0, \gamma_0, \tau(e\phi_0))$ (*not* simply an isomorphism class). Thus we can write

$$D(\phi_0, \gamma_0) = \coprod_{(a',[b'_0])} D_{(a',[b'_0])}$$

where $(a', [b'_0])$ runs over all pairs for which $(\gamma_0, a', [b'_0])$ forms a Kottwitz parameter isomorphic to $(\gamma_0, a, [b_0])$, and $D_{(a',[b'_0])} \subset D(\phi_0, \gamma_0)$ is the subset of twists giving rise to the Kottwitz parameter $(\gamma_0, a', [b_0])$. Note that twisting by $D(\phi_0, \gamma_0)$ will always result in a Kottwitz parameter which: has the same γ_0 (since twisting does not change γ_0); is acceptable and **b**-admissible (since by definition it is isomorphic to our original Kottwitz parameter $(\gamma_0, a, [b_0])$ which is acceptable and **b**-admissible); and has trivial invariant (since it arises from an LR pair).

If $D_{(a',[b'_0])}$ is non-empty, then by [KSZ21, Prop 5.2.4] it is a coset of $Ш_G(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ inside $Ш^\infty_G(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ (note that that proposition only uses their "p^n-admissible" hypothesis to show that $[b_0]$ is basic, which we know by our "acceptable" hypothesis).

To complete the proof, we want to show that $D_{(a',[b'_0])}$ is non-empty. Let $u \in G(\overline{\mathbb{Q}})$ be an element furnishing an isomorphism $(\gamma_0, a, [b_0]) \xrightarrow{\sim} (\gamma_0, a', [b'_0])$. Then $u \in G_{\gamma_0}(\overline{\mathbb{Q}})$ since the two parameters have the same γ_0, and $u\rho(u)^{-1} \in G^\circ_{\gamma_0}$ by the definition of an isomorphism of Kottwitz parameters.

By Lemma 4.5.9, the pair $(\mathrm{Int}(u) \circ \phi_0, \gamma_0)$ is again a gg acceptable $\mathbf{b}$-admissible LR pair ($\mathbf{b}$-admissible because it is conjugate to a $\mathbf{b}$-admissible pair). By Lemma 4.3.15, it has the expected Kottwitz parameter

$$\mathbf{t}(\mathrm{Int}(u) \circ \phi_0, \gamma_0, \tau(\mathrm{Int}(u) \circ \phi_0)) = (\gamma_0, a', [b'_0]).$$

We show that $(\mathrm{Int}(u) \circ \phi_0, \gamma_0)$ is related to (ϕ_0, γ_0) by a twist $e \in Z^1(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ (which is then automatically in $Ш^\infty_G(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ by Proposition 4.5.12). For $\rho \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ let $q_\rho \in \mathfrak{Q}$ a lift of ρ, write $\phi_0(q_\rho) = g_\rho \rtimes \rho$, and define

$$e_\rho = u g_\rho \rho(u)^{-1} g_\rho^{-1}.$$

This defines a cocycle in $Z^1(\mathbb{Q}, I^\circ_{\phi_0,\gamma_0})$ because, working in $I_{\phi_0,\gamma_0}/I^\circ_{\phi_0,\gamma_0}$ (which is an abelian group by [KSZ21, Lemma 5.3.7]), we have seen that g_ρ is trivial and $u\rho(u)^{-1}$ is trivial, which implies e_ρ is trivial. By inspection we have $e\phi_0 = \mathrm{Int}(u) \circ \phi_0$. Thus we have produced an element $e \in D_{(a',[b'_0])}$, which shows that $D_{(a',[b'_0])}$ is non-empty and completes the proof. □

4.5.16 Now we are ready to rewrite our preliminary formula 4.5.1 in terms of Kottwitz parameters. For cleanliness we abbreviate

$$\mathcal{LRP}^\dagger = \{\text{conjugacy classes of acceptable } \mathbf{b}\text{-admissible LR pairs}\},$$

$$\mathcal{KP}^\dagger = \left\{\begin{matrix}\text{isomorphism classes of acceptable } \mathbf{b}\text{-admissible Kottwitz parameters with} \\ \text{trivial Kottwitz invariant}\end{matrix}\right\},$$

$$\Sigma_{\mathbb{R}-\mathrm{ell}}(G) = \{\text{stable conjugacy classes in } G(\mathbb{Q}), \text{ semi-simple and elliptic in } G(\mathbb{R})\},$$

$$\mathcal{KP}(\gamma_0) = \left\{\begin{matrix}\text{pairs } (a, [b_0]) \text{ for which } (\gamma_0, a, [b_0]) \text{ is an acceptable } \mathbf{b}\text{-admissible Kottwitz} \\ \text{parameter with trivial Kottwitz invariant}\end{matrix}\right\}.$$

We argue as follows. Let $f \in C^\infty_c(G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p))$ be an acceptable function, and $\tau \in \Gamma(\mathcal{H})_0$ a tori-rational element satisfying Theorem 3.6.2, which we may lift to a tori-rational element of $\Gamma(\mathfrak{E}^p)_0$ (still called τ by abuse of notation).

$$\begin{aligned}
&\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) \\
&\overset{(1)}{=} \sum_{(\phi,\varepsilon)\in\mathcal{LRP}^\dagger} \frac{1}{[I_{\phi,\varepsilon}(\mathbb{Q}) : I^\circ_{\phi,\varepsilon}(\mathbb{Q})]} T^f_\xi(\phi,\varepsilon,\tau(\phi)) \\
&\overset{(2)}{=} \sum_{(\gamma_0,a,[b_0])\in\mathcal{KP}^\dagger} \sum_{e\in D(\phi_0,\gamma_0)} \frac{1}{\#\{\text{fiber of } \eta_{\phi_0,\gamma_0} \text{ containing } e\}} \frac{1}{[I_{e\phi_0,\gamma_0}(\mathbb{Q}) : I^\circ_{e\phi_0,\gamma_0}(\mathbb{Q})]} T^f_\xi(e\phi_0,\gamma_0,\tau(e\phi_0)) \\
&\overset{(3)}{=} \sum_{(\gamma_0,a,[b_0])\in\mathcal{KP}^\dagger} \sum_{e\in D(\phi_0,\gamma_0)} \frac{1}{|(G_{\gamma_0}/G^\circ_{\gamma_0})(\mathbb{Q})|} T^f_\xi(\gamma_0,a,[b_0]) \\
&\overset{(4)}{=} \sum_{\gamma_0\in\Sigma_{\mathbb{R}-\mathrm{ell}}(G)} \sum_{(a,[b_0])\in\mathcal{KP}(\gamma_0)} \frac{|\text{Ш}_G(\mathbb{Q},G^\circ_{\gamma_0})|}{|(G_{\gamma_0}/G^\circ_{\gamma_0})(\mathbb{Q})|} T^f_\xi(\gamma_0,a,[b_0])
\end{aligned}$$

We explain the essential ideas in each step.

Equality (1) is the preliminary formula 4.5.1 rewritten using the notation of 4.5.6.

Equality (2) uses the results of §4.4, summarized in 4.4.12, to rewrite the sum in terms of isomorphism classes of acceptable **b**-admissible Kottwitz parameters with trivial Kottwitz invariant. In the summand we adjust to this change by summing over $D(\phi_0,\gamma_0)$ which accounts for the fiber of the map **t** over $(\gamma_0, a, [b_0])$, and then dividing by the possible overcounting, as discussed in 4.5.13.

Equality (3) invokes Lemma 4.5.7 to translate the T^f_ξ term from LR pairs to the Kottwitz parameters, as well as Lemma 4.5.14 to rewrite the cardinality of the fiber of η_{ϕ_0,γ_0}.

Equality (4) rewrites the sum over $D(\phi_0,\gamma_0)$ using Lemma 4.5.15. This adds the Ш_G term to the summand, as well as changing the form of the sum. To be precise, substituting Lemma 4.5.15 results in a sum over isomorphism classes of Kottwitz parameters $(\gamma_0, a, [b_0])$ followed by a sum over pairs $(a', [b'_0])$ for which $(\gamma_0, a', [b'_0])$ is a Kottwitz parameter isomorphic to $(\gamma_0, a, [b_0])$—we replace this with a sum over stable conjugacy classes in $G(\mathbb{Q})$ which are semi-simple and $\mathbb{R}$-elliptic, followed by a sum over pairs $(a, [b_0])$ for which $(\gamma_0, a, [b_0])$ is an acceptable **b**-admissible Kottwitz parameter with trivial invariant.

Unwinding once again the notation of 4.5.6, we arrive at the final form of our unstable trace formula for Igusa varieties of Hodge type.

Theorem 4.5.17. *For any acceptable function $f \in C^\infty_c(G(\mathbb{A}^p_f) \times J_b(\mathbb{Q}_p))$, we have*

$$\begin{aligned}
&\operatorname{tr}(f \mid H_c(\mathrm{Ig}_\Sigma, \mathscr{L}_\xi)) = \\
&\sum_{\gamma_0\in\Sigma_{\mathbb{R}-\mathrm{ell}}(G)} \sum_{(a,[b_0])\in\mathcal{KP}(\gamma_0)} \frac{|\text{Ш}_G(\mathbb{Q},G^\circ_{\gamma_0})|}{|(G_{\gamma_0}/G^\circ_{\gamma_0})(\mathbb{Q})|} \operatorname{vol}\left(I^\circ_{\mathfrak{c}}(\mathbb{Q})\backslash I^\circ_{\mathfrak{c}}(\mathbb{A}_f)\right) O^{G(\mathbb{A}^p_f)\times J_b(\mathbb{Q}_p)}_{\gamma\times\delta}(f)\operatorname{tr}(\xi(\gamma_0))
\end{aligned}$$

where $I_{\mathfrak{c}}$ is the inner form of $G^{\circ}_{\gamma_0}$ associated to the Kottwitz parameter $\mathfrak{c} = (\gamma_0, a, [b_0])$ as in 4.5.3, and γ, δ are the elements belonging to the classical Kottwitz parameter $(\gamma_0, \gamma, \delta)$ associated to $\mathfrak{c}$ as in 4.3.18.

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