Problem Set 9 Solutions

MATH 16B Spring 2016

28 April 2015

Exercise. For the following continuous random variables (given in terms of their probability distribution function), calculate the probability given.

(a) $f(x) = 3x^2$, $0 \le x \le 1$. What is $P(0 \le X \le \frac{1}{2})$?

(b) $f(x) = \frac{\sin x}{2}$, $0 \le x \le \pi$. What is $P(\frac{\pi}{3} \le X \le \pi)$?

Solution. Recall that

$$P(a \le X \le b) = \int_a^b f(x)dx.$$

(a) This probability is

$$\int_0^{1/2} 3x^2 dx = x^3\Big|_0^{1/2} = \frac{1}{8}.$$

(b) This probability is

$$\int_{\pi/3}^{\pi} \frac{\sin x}{2} dx = -\frac{\cos x}{2}\Big|_{\pi/3}^{\pi} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

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Exercise. Find the expected value and variance of each of the following continuous random variables (given in terms of their probability density function).

(a) $f(x) = 5e^{-5x}$, $x \ge 0$.

(b) $f(x) = 12x^2(1-x)$, $0 \le x \le 1$.

Solution. Recall that the expected value and variance of a random variable are given by

$$\mathrm{E}(X) = \int_A^B x f(x) dx$$

and

$$\mathrm{Var}(X) = \int_A^B x^2 f(x) dx - \mathrm{E}(X)^2.$$

(a) The expected value is

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^\infty 5xe^{-5x}dx \\
&= -xe^{-5x}\Big|_0^\infty + \int_0^\infty e^{-5x}dx \\
&= \left[-xe^{-5x} - \frac{1}{5}e^{-5x}\right]_0^\infty \\
&= \frac{1}{5}.
\end{aligned}$$

Here we evaluated the integral using integration by parts (with $u = x$, $dv = 5e^{-5x}dx$). Note also that $-xe^{-5x}$ and $-\frac{1}{5}e^{-5x}$ both go to 0 as $x \to \infty$.

The variance is

$$\begin{aligned}
\mathrm{Var}(X) &= \int_0^\infty 5x^2e^{-5x}dx - \frac{1}{25} \\
&= -x^2e^{-5x}\Big|_0^\infty + \int_0^\infty 2e^{-5x}dx - \frac{1}{25} \\
&= -x^2e^{-5x}\Big|_0^\infty + \frac{2}{5}\int_0^\infty 5e^{-5x}dx - \frac{1}{25} \\
&= \frac{2}{25} - \frac{1}{25} \\
&= \frac{1}{25}.
\end{aligned}$$

Here we used integration by parts (with $u = x^2$, $dv = 5e^{-5x}$) and then manipulated the remaining integral to make it exactly the same as the integral for expected value above (for which we can then simply substitute in the value we found for that integral). Note also that $-x^2e^{-5x}$ is zero both at $x = 0$ and as $x \to \infty$.

(b) The expected value is

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^1 12x^3(1-x)dx \\
&= \int_0^1 12x^3 - 12x^4dx \\
&= \left[3x^4 - \frac{12}{5}x^5\right]_0^1 \\
&= 3 - \frac{12}{5} \\
&= \frac{3}{5}.
\end{aligned}$$

The variance is

$$\begin{aligned}
\text{Var}(X) &= \int_0^1 12x^4(1-x)dx - \frac{9}{25} \\
&= \int_0^1 12x^4 - 12x^5 dx - \frac{9}{25} \\
&= \left[\frac{12}{5}x^5 - 2x^6\right]_0^1 - \frac{9}{25} \\
&= \frac{12}{5} - 2 - \frac{9}{25} \\
&= \frac{1}{25}.
\end{aligned}$$

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