Problem Set 8 Solutions

MATH 16B Spring 2016

21 April 2015

Exercise. Decide whether the following sums converge or diverge.

(a)

$$\sum_{n=1}^{\infty} \frac{1}{n^3+n+1}$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$$

Solution. (a) We compare this series to $\sum \frac{1}{n^3}$. Note that

$$0 \le \frac{1}{n^3+n+1} \le \frac{1}{n^3}$$

for all $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is a p-series with $p = 3$. By the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^3+n+1}$ converges also.

(b) We compare to $\sum \frac{1}{n^2}$. Note that

$$0 \le \frac{1}{n^2 \ln(n)} \le \frac{1}{n^2}$$

for all $n \ge 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it is a p-series with $p = 2$. By the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2 \ln(n)}$ converges also. $\square$

Exercise. Compute the Taylor series of the following functions at $x = 0$. You may use the Taylor series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. (Hint: you may have to take a derivative for one of these).

(a)

$$xe^{x^2}$$

(b)

$$\frac{1}{(1+x)^2}$$

Solution. (a) The function xe^{x^2} can be made from e^x by replacing x by x^2 and then multiplying by x. Applying these manipulations to the Taylor series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we find that the Taylor series of xe^{x^2} is

$$xe^{x^2} = \sum_{n=0}^{\infty} x \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}.$$

(b) The derivative of $\frac{1}{1-x}$ is $\frac{1}{(1-x)^2}$. We can get the function $\frac{1}{(1+x)^2}$ from this derivative by replacing x with $-x$. Taking the derivative and then replacing x by $-x$ in the Taylor series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we find that the Taylor series of $\frac{1}{(1+x)^2}$ is

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} n(-x)^{n-1} = \sum_{n=0}^{\infty} n(-1)^{n-1}x^{n-1}.$$

□

Exercise. Find the expected value, variance and standard deviation of the following discrete random variable.

Outcome	1	2	3
Probability	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Solution. The expected value is the sum of the outcomes weighted by their probability:

$$\frac{4}{9} \cdot 1 + \frac{4}{9} \cdot 2 + \frac{1}{9} \cdot 3 = \frac{5}{3}.$$

The variance is the sum of squared differences from the expected value, again weighted by probability:

$$\frac{4}{9}\left(1 - \frac{5}{3}\right)^2 + \frac{4}{9}\left(2 - \frac{5}{3}\right)^2 + \frac{1}{9}\left(3 - \frac{5}{3}\right)^2 = \frac{4}{9}.$$

The standard deviation is simply the square root of the variance:

$$\sqrt{\frac{4}{9}} = \frac{2}{3}.$$

□