Problem Set 7 Solutions

MATH 16B Spring 2016

14 April 2015

Decide whether the sum converges or diverges. If it is a convergent geometric series, find the sum.

Exercise.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Solution. We'll use the integral test. Note that $f(x) = \frac{1}{x^3}$ is positive, decreasing, and continuous for $x \geq 1$. Also

$$\int_1^{\infty} \frac{1}{x^3} dx = -\frac{1}{2x^2}\bigg|_1^{\infty} = \lim_{a \to \infty} -\frac{1}{2a^2} + \frac{1}{2} = \frac{1}{2}.$$

Since the integral is convergent, the integral test tells us that the sum is convergent as well. □

Exercise.

$$\sum_{n=1}^{\infty} \frac{5 \cdot 3^{n/2}}{2^n}$$

Solution. This is a geometric series with $a = \frac{5\sqrt{3}}{2}$ and $r = \frac{\sqrt{3}}{2}$. Since $|r| < 1$ this series is convergent, and its sum is

$$\frac{5\sqrt{3}}{2} \frac{1}{1 - \frac{\sqrt{3}}{2}} = \frac{5\sqrt{3}}{2 - \sqrt{3}}.$$

□

Exercise.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^2}$$

Solution. We'll use the integral test. Note that $f(x) = \frac{1}{x \ln(x)^2}$ is positive, decreasing, and continuous for $x \geq 2$. Also (using the substitution $u = \ln(x)$ and $du = \frac{1}{x} dx$),

$$\int_2^{\infty} \frac{1}{x \ln(x)^2} dx = \int_{x=2}^{x=\infty} \frac{1}{u^2} du = -\frac{1}{u}\bigg|_{x=2}^{x=\infty} = -\frac{1}{\ln(x)}\bigg|_2^{\infty} = \lim_{a \to \infty} -\frac{1}{\ln(a)} + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}.$$

Thus the integral converges, and by the integral test the sum converges as well. □

Exercise.

$$\sum_{n=1}^{\infty} e^{-n}$$

Solution. This is a geometric series with $a = \frac{1}{e}$ and $r = \frac{1}{e}$. Since $|r| < 1$ this series converges, and the sum is

$$\frac{1/e}{1 - 1/e} = \frac{1}{e - 1}.$$

□

Exercise.

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Solution. We'll use the integral test. Let $f(x) = \frac{x}{2^x} = x2^{-x}$. Then

$$f'(x) = 2^{-x} - x\ln(2)2^{-x} = (1 - x\ln(2))2^{-x},$$

so $f(x)$ is decreasing (and also positive and continuous) for $x \geq 2$. Also, using integration by parts with

$$u = x, \qquad du = dx, \qquad v = -\frac{2^{-x}}{\ln(2)}, \qquad dv = 2^{-x}dx$$

we find

$$\begin{aligned}
\int_2^{\infty} x2^{-x}dx &= -\frac{x2^{-x}}{\ln(2)}\Big|_2^{\infty} + \int_2^{\infty} \frac{2^{-x}}{\ln(2)}dx \\
&= \left[-\frac{x2^{-x}}{\ln(2)} - \frac{2^{-x}}{\ln(2)^2}\right]_2^{\infty} \\
&= \lim_{a\to\infty}\left(-\frac{a}{2^a\ln(2)} - \frac{1}{2^a\ln(2)^2}\right) - \left(-\frac{1}{2\ln(2)} - \frac{1}{4\ln(2)^2}\right) \\
&= \frac{1}{2\ln(2)} + \frac{1}{4\ln(2)^2}.
\end{aligned}$$

Thus the integral converges, and by the integral test the sum converges as well. □