Problem Set 6 Solutions

MATH 16B Spring 2016

7 April 2015

Exercise. Find the 3^{rd} order Taylor poynomial of $f(x) = \sqrt{x}$ at $x = 1$. Use it to approximate $\sqrt{2}$, and give a bound for the error in this approximation.

Solution. The first three derivatives of $f(x) = \sqrt{x}$ are

$$f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}$$

and evaluating these at $x = 1$ where we're setting the Taylor polynomial,

$$f'(1) = \frac{1}{2}, \qquad f''(1) = -\frac{1}{4}, \qquad f'''(1) = \frac{3}{8}.$$

So the third order Taylor polynomial is

$$p_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

We can use this to estimate $\sqrt{2}$; namely, our estimate is

$$p_3(2) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = 23/16.$$

To find a bound for the error we use the remainder formula. The main idea is to find a bound for the absolute value of the 4^{th} derivative

$$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$$

on the interval from 1 to 2. Since $-\frac{15}{16}x^{-7/2}$ is decreasing in absolute value as x increases from 1 to 2, the maximum absolute value occurs at 1, where the value is $-\frac{15}{16}$. Thus

$$|f^{(4)}(x)| = \left| -\frac{15}{16}x^{-7/2} \right| \leq \frac{15}{16}$$

for x between 1 and 2. Now using the remainder formula, with $M = \frac{15}{16}$, we find

$$|\sqrt{2} - \tfrac{23}{16}| \leq \frac{15}{16}\frac{1}{4!}|2-1|^4 = \frac{5}{128}.$$

That is, the error in our approximation is at most $\frac{5}{128}$. □

Exercise. Find a formula for the n^{th} order Taylor polynomial of e^x at $x = 0$.

Solution. The n^{th} derivative of $f(x) = e^x$ is e^x for any n, and evaluating at $x = 0$ we find $f^{(n)}(0) = e^0 = 1$ for all n. Thus the n^{th} order Taylor polynomial of e^x at $x = 0$ is

$$p_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(x)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

□

Exercise (11.3.5, 11.3.11)**.** Decide whether the sum converges or diverges, and find the value if it converges.

- $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} \cdots$
- $\frac{2}{5^4} - \frac{2^4}{5^5} + \frac{2^7}{5^6} - \frac{2^{10}}{5^7} + \cdots$

Solution. The first is a geometric series with $a = 2$ and $r = \frac{1}{3}$. Since $|r| < 1$ this series converges, and the sum is

$$\frac{a}{1-r} = \frac{2}{1 - \frac{1}{3}} = 3.$$

The second is a geometric series with $a = \frac{2}{5^4}$ and $r = -\frac{2^3}{5}$. Since $|r| \geq 1$ this series diverges. □