Problem Set 2 Solutions

MATH 16B Spring 2016

11 February 2015

Exercise (7.3.26)**.** Find all maxima and minima of the function

$$f(x,y) = x^2 + 4xy + 2y^4.$$

Solution. The first step is to apply the first derivative test to find all possible locations of maxima and minima.

$$\frac{\partial f}{\partial x} = 2x + 4y = 0, \qquad \frac{\partial f}{\partial y} = 4x + 8y^3 = 0.$$

The first equation gives us $x = -2y$, and plugging this into the second equation gives

$$0 = 4(-2y) + 8y^3 = 8y^3 - 8y = 8y(y+1)(y-1).$$

The solutions to this equation are $y = 0, 1, -1$, and using $x = -2y$ we find three possible max/min points: $(0,0)$, $(1,-2)$, and $(-1,2)$.

Now we test these points using the second derivative test.

$$\frac{\partial^2 f}{\partial x^2} = 2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 4, \qquad \frac{\partial^2 f}{\partial y^2} = 24y^2$$

We find

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 48y^2 - 16.$$

At $(0,0)$: $D < 0$, so this is neither a max nor min.
At $(1,-2)$: $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so this is a min.
At $(-1,2)$: $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so this is a min. □

Exercise (7.3.27)**.** Find all possible points where

$$f(x,y,z) = 2x^2 + 3y^2 + z^2 - 2x - y - z$$

could have a maximum or minimum.

Solution. To do this we use the first derivative test.

$$\frac{\partial f}{\partial x} = 4x - 2 = 0, \qquad \frac{\partial f}{\partial y} = 6y - 1 = 0, \qquad \frac{\partial f}{\partial z} = 2z - 1 = 0.$$

Since each equation involves only one variable, we can solve to find $x = \frac{1}{2}$, $y = \frac{1}{6}$, and $z = \frac{1}{2}$. Thus the only possible point where f could have a max or min is $(\frac{1}{2}, \frac{1}{6}, \frac{1}{2})$. □

Exercise (7.4.6)**.** Minimize
$$x^2 + xy + y^2 - 2x - 5y$$
subject to the constraint
$$1 - x + y = 0.$$
(Note: you do not have to verify that the point you find is indeed a minimum).

Solution. This problem asks us to maximize a function with a constraint, so we use Lagrange multipliers. Define

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = x^2 + xy + y^2 - 2x - 5y + \lambda(1 - x + y),$$

where $f(x, y) = x^2 + xy + y^2 - 2x - 5y$ is the function we are trying to maximize and $g(x, y) = 1 - x + y = 0$ is the constraint.

To minimize $f(x, y)$ we find the points where all partial derivatives of $F(x, y, \lambda)$ are zero.

$$\frac{\partial F}{\partial x} = 2x + y - 2 - \lambda = 0, \qquad \frac{\partial F}{\partial y} = x + 2y - 5 + \lambda = 0, \qquad \frac{\partial F}{\partial \lambda} = 1 - x + y = 0.$$

Now we just want to solve this system of linear equations. Adding the first two equations together gives the equation $3x + 3y - 7 = 0$, and substituting $x = y + 1$ (from the third equation above) gives $6y - 4 = 0$. We find $y = \frac{2}{3}$ and $x = \frac{5}{3}$. Thus the function f is minimized with respect to the constraint $g = 0$ at the point $(\frac{5}{3}, \frac{2}{3})$. □

Exercise (7.4.19)**.** Find the values of x, y, z that maximize

$$h(x, y, z) = 3x + 5y + z - x^2 - y^2 - z^2$$

subject to the constraint
$$g(x, y, z) = x + y + z = -6.$$
(Note: you do not have to verify that the point you find is indeed a maximum).

Solution. Lagrange multipliers again! Define

$$F(x, y, z, \lambda) = 3x + 5y + z - x^2 - y^2 - z^2 + \lambda(x + y + z + 6)$$

(note that the constraint is $x + y + z + 6 = 0$, so this is the thing we multiply by λ). Now check where all partial derivatives of $F(x, y, z, \lambda)$ are zero:

$$\frac{\partial F}{\partial x} = 3 - 2x + \lambda, \quad \frac{\partial F}{\partial y} = 5 - 2y + \lambda, \quad \frac{\partial F}{\partial z} = 1 - 2z + \lambda, \quad \frac{\partial F}{\partial \lambda} = x + y + z + 6.$$

Solving for the other variable in terms of λ in each of the first three equations gives

$$x = \frac{1}{2}(3 + \lambda), \qquad y = \frac{1}{2}(5 + \lambda), \qquad z = \frac{1}{2}(1 + \lambda).$$

Substituting these into the last equation and solving for λ gives $\lambda = -7$, so we find $x = -2$, $y = -1$, and $z = -3$. Thus the maximum of $h(x, y, z)$ with respect to the constraint is achieved at $(-2, -1, -3)$. □

Exercise. State precisely

- the first derivative test for a function of two variables, and
- the second derivative test for a function of two variables.

Solution. The first derivative test for a function of two variables states: If a function $f(x, y)$ of two variables has a minimum or maximum at a point (a, b), then $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$.

The second derivative test for a function of two variables is as follows. Let $f(x, y)$ be a function of two variables and suppose (a, b) is a point with $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$. Define

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2.$$

Then:

- If $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ then (a, b) is a minimum.
- If $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ then (a, b) is a maximum.
- If $D < 0$ then (a, b) is neither a maximum nor a minimum.
- If $D = 0$ then the test is inconclusive.

□