Midterm 2 Review Solutions

MATH 16B Spring 2016

Exercise 1. Compute

$$\int_0^{\sqrt{\pi}} x\sin(x^2)dx \qquad \text{and} \qquad \int_0^{\pi} x^2 \sin x\, dx.$$

Solution. For the first we use substitution, and for the second we use integration by parts.

The substitution in the first is $u = x^2$, so $du = 2xdx$. When $x = 0$ we have $u = 0$, and when $x = \sqrt{\pi}$ we have $u = \pi$. Thus

$$\begin{aligned}\int_0^{\sqrt{\pi}} x\sin(x^2)dx &= \frac{1}{2}\int_0^{\pi} \sin u\, du \\ &= \left[-\frac{1}{2}\cos u\right]_0^{\pi} \\ &= -\frac{1}{2}\cdot(-1) + \frac{1}{2}\cdot 1 \\ &= 1.\end{aligned}$$

For the second we must integrate by parts twice. Both times we set u to be the x term outside the trig function, and set dv to be the rest. So the first will be

$$u = x^2, \quad du = 2xdx, \quad v = -\cos x, \quad dv = \sin x\, dx$$

(the second I won't spell out). Using this, we find

$$\begin{aligned}\int_0^{\pi} x^2 \sin x\, dx &= \left[-x^2\cos x\right]_0^{\pi} + \int_0^{\pi} 2x\cos x\, dx \\ &= \left[-x^2\cos x\right]_0^{\pi} + [2x\sin x]_0^{\pi} - \int_0^{\pi} 2\sin x\, dx \\ &= \left[-x^2\cos x\right]_0^{\pi} + [2x\sin x]_0^{\pi} - [-2\cos x]_0^{\pi} \\ &= \pi^2 - 4.\end{aligned}$$

□

Exercise 2. Find all solutions to the differential equation

$$y' = -(y+1)^2(t+1)$$

(including possibly constant solutions).

Solution. For this differential equation we can use separation of variables.

$$\begin{aligned}
y' &= -(y+1)^2(t+1) \\
\frac{1}{(y+1)^2}\frac{dy}{dt} &= -t-1 \\
\int \frac{1}{(y+1)^2}dy &= \int -t-1dt \\
-\frac{1}{y+1} &= -\frac{1}{2}t^2 - t + c \\
y+1 &= \frac{1}{\frac{1}{2}t^2+t+c} \\
y &= \frac{1}{\frac{1}{2}t^2+t+c} - 1.
\end{aligned}$$

This general solution doesn't produce any constant solutions, so we should check for those as well. If we have a constant solution then the left hand side (i.e. y') of our differential equation will be 0, so the right hand side must be 0 as well, and to get this we can set $y = -1$. So $y = -1$ is a constant solution. □

Exercise 3. Compute

$$\int (\ln x)^2 dx.$$

Solution. We should use integration by parts (twice) to get rid of the log by differentiating it. For the first integration by parts, set

$$u = (\ln x)^2, \quad du = \frac{2}{x}\ln x, \quad v = x, \quad dv = dx$$

(I won't spell out the second one). Now using integration by parts we find

$$\begin{aligned}
\int (\ln x)^2 dx &= x(\ln x)^2 - \int 2\ln x\, dx \\
&= x(\ln x)^2 - 2x\ln x + \int 2dx \\
&= x(\ln x)^2 - 2x\ln x + 2x.
\end{aligned}$$

□

Exercise 4. Compute

$$\int x(3x^2+1)^5 dx.$$

Solution. This is just a polynomial, so we can integrate it straight up, but it'd be easier to use substitution (to avoid expanding out the fifth power). Set $u = 3x^2+1$, so $du = 6xdx$. We find

$$\begin{aligned}
\int x(3x^2+1)^5 dx &= \frac{1}{6}\int u^5 du \\
&= \frac{1}{36}u^6 + c \\
&= \frac{1}{36}(3x^2+1)^6 + c.
\end{aligned}$$

□

Exercise 5. Solve the following initial value problem.

$$y' + 2y\cos(2t) = 2\cos 2t, \quad y(\pi/2) = 0.$$

Solution. We can use either separation of variables or integrating factors for this differential equation, but let's use integrating factors.

This is a first-order linear differential equation (i.e. $y' + a(t)y = b(t)$) with

$$a(t) = 2\cos 2t, \qquad b(t) = 2\cos 2t.$$

An antiderivative of $a(t)$ is $A(t) = \sin 2t$. The integrating factors method tells us a general solution is given by

$$y = e^{-A(t)}\left[\int e^{A(t)}b(t)dt + c\right],$$

so in this case,

$$\begin{aligned} y &= e^{-\sin 2t}\left[\int e^{\sin 2t} 2\cos 2t\, dt + c\right] \\ &= e^{-\sin 2t}\left[\int e^{u} du + c\right] \\ &= e^{-\sin 2t}\left[e^{u} + c\right] \\ &= e^{-\sin 2t}\left[e^{\sin 2t} + c\right] \\ &= 1 + ce^{-\sin 2t}. \end{aligned}$$

(where we've computed the integral using the substitution $u = \sin 2t$).

We also want $y(\pi/2) = 0$, so plugging $t = \pi/2$ and $y = 0$ into our general solution we find

$$0 = 1 + ce^{0}$$

and so $c = -1$. Thus the solution to the initial value problem is

$$y = 1 - e^{-\sin 2t}.$$

□

Exercise 6. Compute

$$\int \tan 2x\, dx.$$

Solution. To compute this integral we first rewrite it as

$$\int \frac{\sin 2x}{\cos 2x} dx.$$

Now we can use substitution with $u = \cos 2x$ and $du = -2\sin 2x\, dx$ to get

$$\begin{aligned} \int \frac{\sin 2x}{\cos 2x} dx &= -\frac{1}{2}\int \frac{1}{u} du \\ &= -\frac{1}{2}\ln u + c \\ &= -\frac{1}{2}\ln(\cos 2x) + c. \end{aligned}$$

□

Exercise 7. Compute

$$\int_{-\infty}^{0} e^{4x} dx.$$

Solution. First we write the improper integral as a limit, and then evaluate the integral with proper bounds, and finally evaluate the limit.

$$\begin{aligned}
\int_{-\infty}^{0} e^{4x} dx &= \lim_{a\to\infty} \int_{-a}^{0} e^{4x} dx \\
&= \lim_{a\to\infty} \left[\frac{1}{4}e^{4x}\right]_{-a}^{0} \\
&= \lim_{a\to\infty} \frac{1}{4} - \frac{1}{4}e^{-4a} \\
&= \frac{1}{4}.
\end{aligned}$$

□

Exercise 8. Solve the following initial value problem.

$$y' + 2y = 1, \quad y(0) = 1.$$

Solution. We could use separation of variables or integrating factors, but let's use integrating factors. This is a first-order linear differential equation (i.e. $y' + a(t)y = b(t)$) with

$$a(t) = 2, \qquad b(t) = 1.$$

An antiderivative of $a(t)$ is $A(t) = 2t$. The integrating factors method tells us a general solution is given by

$$y = e^{-A(t)} \left[\int e^{A(t)} b(t) dt + c\right],$$

so in this case,

$$\begin{aligned}
y &= e^{-2t} \left[\int e^{2t} dt + c\right] \\
&= e^{-2t} \left[\frac{1}{2}e^{2t} + c\right] \\
&= \frac{1}{2} + ce^{-2t}.
\end{aligned}$$

We also want $y(0) = 1$, so plugging $t = 0$ and $y = 1$ into this equation we find

$$1 = \frac{1}{2} + ce^{0},$$

so $c = \frac{1}{2}$. Thus the solution to the initial value problem is

$$y = \frac{1}{2} + \frac{1}{2}e^{-2t}.$$

□

Exercise 9. Compute

$$\int_{0}^{\infty} xe^{-x^2} dx.$$

Solution. First we write the improper integral as a limit, and then evaluate the integral with proper bounds, and finally evaluate the limit.

$$\begin{aligned}
\int_0^\infty xe^{-x^2}dx &= \lim_{a\to\infty}\int_0^a xe^{-x^2}dx \\
&= \lim_{a\to\infty}\int_{x=0}^{x=a} -\frac{1}{2}e^u du \\
&= \lim_{a\to\infty} -\frac{1}{2}e^u\Big|_{x=0}^{x=a} \\
&= \lim_{a\to\infty} -\frac{1}{2}e^{-x^2}\Big|_{x=0}^{x=a} \\
&= \lim_{a\to\infty} -\frac{1}{2}e^{-a^2}+\frac{1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

(where we evaluated the integral using the substitution $u=-x^2$). □

Exercise 10. Solve the following initial value problem.

$$y' = \frac{\ln x}{\sqrt{xy}}, \quad y(1)=4.$$

Solution. We use separation of variables.

$$\begin{aligned}
y' &= \frac{\ln x}{\sqrt{xy}} \\
\sqrt{y}\frac{dy}{dt} &= \frac{\ln x}{\sqrt{x}} \\
\int \sqrt{y}dy &= \int \frac{\ln x}{\sqrt{x}}dt \\
\frac{2}{3}y^{3/2} &= 2\sqrt{x}\ln x - \int \frac{2}{\sqrt{x}}dx \\
\frac{2}{3}y^{3/2} &= 2\sqrt{x}\ln x + 4\sqrt{x}+c \\
y^{3/2} &= 3\sqrt{x}\ln x + 6\sqrt{x}+c \\
y &= \left(3\sqrt{x}\ln x + 6\sqrt{x}+c\right)^{2/3}.
\end{aligned}$$

(where we evaluated the integral using integration by parts with $u=\ln x$ and $dv=\frac{1}{\sqrt{x}}dx$).

We also want $y(1)=4$, so plugging in $x=1$ and $y=4$

$$4 = (3\ln 1 + 6 + c)^{2/3}$$

we find $c=2$. Thus the solution to the initial value problem is

$$y = \left(3\sqrt{x}\ln x + 6\sqrt{x}+2\right)^{2/3}.$$

□