Midterm 1 Review Solutions

MATH 16B Spring 2016

Exercise 1. Compute

$$\int_2^5 \int_0^1 9x^2y^2 dy\, dx$$

Solution. Using the power rule,

$$\begin{aligned}\int_2^5 \int_0^1 9x^2y^2 dy\, dx &= \int_2^5 \left[3x^2y^3\right]_{y=0}^1 dx \\ &= \int_2^5 3x^2 dx \\ &= \left[x^3\right]_{x=2}^5 \\ &= 117.\end{aligned}$$

□

Exercise 2. Let D be the region of the x, y-plane where x and y are greater than or equal to zero and their sum is at most 3. Find the volume of the solid bounded above by $f(x, y) = e^x$ and lying over D.

Solution. The lines bounding D are $x = 0$, $y = 0$, and $x + y = 3$. Integrating with respect to x first (no reason really to choose x over y), our limits should be $x = 0$ below and $x = -y + 3$ above; then the y values should range from 0 to 3. Thus the desired volume is

$$\begin{aligned}\int_0^3 \int_0^{-y+3} e^x dx\, dy &= \int_0^3 \left[e^x\right]_{x=0}^{-y+3} dy \\ &= \int_0^3 e^{-y+3} - 1\, dy \\ &= \left[-e^{-y+3} - y\right]_{y=0}^3 \\ &= -4 + e^3.\end{aligned}$$

□

Exercise 3. Find the maximum value of

$$h(x, y) = x + 2y - x^2 + xy - y^2.$$

(Note that this asks for the maximum value, not the location where the maximum occurs).

Solution. Applying the first derivative test,

$$\frac{\partial h}{\partial x} = 1 - 2x + y = 0, \qquad \frac{\partial h}{\partial y} = 2 + x - 2y = 0.$$

The first equation tells us $y = 2x - 1$; plugging this into the second equation gives $x = \frac{4}{3}$, so $y = \frac{5}{3}$. Thus the only point to check is $(\frac{4}{3}, \frac{5}{3})$.

Now the second derivative test: we compute

$$\frac{\partial^2 h}{\partial x^2} = -2, \qquad \frac{\partial^2 h}{\partial x \partial y} = 1, \qquad \frac{\partial^2 h}{\partial y^2} = -2, \qquad D = 3.$$

Since $D > 0$ and $\frac{\partial^2 h}{\partial x^2} < 0$, this point is indeed a maximum. The value at this point is $h(\frac{4}{3}, \frac{5}{2}) = \frac{7}{3}$, so $\frac{7}{3}$ is the maximum value of h. □

Exercise 4. Compute $\frac{\partial^2 g}{\partial x \partial y}$ and $\frac{\partial^2 g}{\partial y \partial x}$ for

$$g(x, y) = x^y$$

and observe that they are equal.

Solution. If we're differentiating with respect to x (the base) we use the power rule, and if we're differentiating with respect to y (the exponent) we use the exponent rule. If we do x first,

$$\frac{\partial^2}{\partial y \partial x} x^y = \frac{\partial}{\partial y} y x^{y-1} = x^{y-1} + y \ln(x) x^{y-1},$$

and if we do y first,

$$\frac{\partial^2}{\partial x \partial y} x^y = \frac{\partial}{\partial x} \ln(x) x^y = \frac{1}{x} x^y + \ln(x) y x^{y-1},$$

and after rearranging slightly we are happy to see that these are the same. □

Exercise 5. Maximize

$$f(x, y, z) = 3x - 3y - 8z - 2x^2 + xy - z^2$$

with respect to the constraint

$$g(x, y, z) = -x + y + 3z = -1.$$

Solution. Using Lagrange multipliers, we define the function

$$F(x, y, z, \lambda) = 3x - 3y - 8z - 2x^2 + xy - z^2 + \lambda(-x + y + 3z + 1)$$

(note we multiply λ by $-x + y + 3z + 1$ because our constraint is $-x + y + 3z + 1 = 0$). Now we compute the first derivatives of F and find where they are all zero:

$$\frac{\partial F}{\partial x} = 3 - 4x + y - \lambda = 0$$
$$\frac{\partial F}{\partial y} = -3 + x + \lambda = 0$$
$$\frac{\partial F}{\partial z} = -8 - 2z + 3\lambda = 0$$
$$\frac{\partial F}{\partial \lambda} = -x + y + 3z + 1 = 0$$

Solving this system of equations we find $x = 1$, $y = 3$, $z = -1$. Thus the maximum occurs at $(1, 3, -1)$, and the maximum value is $f(1, 3, -1) = 2$. □

Exercise 6. Compute

$$\int_0^{\pi/4} \int_x^{2x} \cos y \, dy \, dx.$$

Solution. Recall that $\int \cos x \, dx = \sin x$ and $\int \sin x \, dx = -\cos x$. So

$$\begin{aligned}
\int_0^{\pi/4} \int_x^{2x} \cos y \, dy \, dx &= \int_0^{\pi/4} [\sin y]_{y=x}^{2x} \, dx \\
&= \int_0^{\pi/4} \sin(2x) - \sin(x) \, dx \\
&= \left[-\frac{1}{2} \cos(2x) + \cos(x) \right]_{x=0}^{\pi/4} \\
&= \left(-\frac{1}{2} \cos(\pi/2) + \cos(\pi/4) \right) - \left(-\frac{1}{2} \cos(0) + \cos(0) \right) \\
&= \frac{\sqrt{2}}{2} - \frac{1}{2}.
\end{aligned}$$

□

Exercise 7. State precisely the first and second derivative tests for functions of two variables.

Solution. The first derivative test for a function $f(x, y)$ of two variables says that if (a, b) is a maximum or minimum for f then $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$.

The second derivative test for a function $f(x, y)$ of two variables is as follows. Let

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2.$$

If (a, b) is a point with $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$, then

- if $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$ then (a, b) is a minimum;
- if $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$ then (a, b) is a maximum;
- if $D(a, b) < 0$ then (a, b) is neither a maximum nor minimum; and
- if $D(a, b) = 0$ then the test is inconclusive.

□

Exercise 8. Find all maxima and minima of

$$f(x, y) = 2x^2 - x^4 - y^2.$$

Solution. The first derivative test:

$$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0, \qquad \frac{\partial f}{\partial y} = -2y = 0.$$

The first equation factors as $-4x(x+1)(x-1)$, so it has three solutions $x = 0, -1, 1$. The second equation has one solution $y = 0$. So we need to check the points $(0,0)$, $(-1,0)$, and $(1,0)$.

The second derivative test prep is:

$$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 0, \qquad \frac{\partial^2 f}{\partial y^2} = -2, \qquad D = -8 + 24x^2.$$

At $(0,0)$: $D < 0$, so neither a max nor min.
At $(-1,0)$: $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, so this is a max,
At $(1,0)$: $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, so this is a max as well. □

Exercise 9. Find all possible points where

$$g(x,y,z) = 3x + 3y - z - x^2 + xy - y^2 - z^2$$

could have a maximum.

Solution. We use the first derivative test.

$$\frac{\partial g}{\partial x} = 3 - 2x + y = 0, \qquad \frac{\partial g}{\partial y} = 3 + x - 2y, \qquad \frac{\partial g}{\partial z} = -1 - 2z = 0.$$

The first two $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ give a system of two equations in the two variables x, y, and solving gives $x = 3$ and $y = 3$. The last equation gives $z = -\frac{1}{2}$. Thus $(3, 3, -\frac{1}{2})$ is the only point where g could have a maximum. □

Exercise 10. Compute all first and second partial derivatives of

$$f(x,y) = \sin x \sin y + \cos 2xy$$

Solution. Remembering the chain rule and the rules $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$, we find

$$\frac{\partial f}{\partial x} = \cos x \sin y - 2y \sin 2xy, \qquad \frac{\partial f}{\partial y} = \sin x \cos y - 2x \sin 2xy,$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x \sin y - 4y^2 \cos 2xy, \qquad \frac{\partial^2 f}{\partial x \partial y} = \cos x \cos y - 2 \sin 2xy - 4xy \cos 2xy,$$

$$\frac{\partial^2 f}{\partial y^2} = -\sin x \sin y - 4x^2 \cos 2xy.$$

□

Exercise 11. The function

$$f(x,y) = 4x + 3y - 1$$

has one maximum and one minimum with respect to the constraint

$$x^2 + y^2 = 25.$$

Find the two points where the maximum and minimum occur.

Solution. To do Lagrange Multipliers, we define

$$F(x, y, \lambda) = 4x + 3y - 1 + \lambda(x^2 + y^2 - 25),$$

and check where all its first derivatives are zero. We get the equations

$$\frac{\partial F}{\partial x} = 4 + 2\lambda x = 0, \qquad \frac{\partial F}{\partial y} = 3 + 2\lambda y = 0, \qquad \frac{\partial F}{\partial \lambda} = x^2 + y^2 - 25 = 0.$$

Note that there is no solution if $\lambda = 0$, so we can assume this is not the case (i.e. we can divide by λ). From the first two equations we get $x = -\frac{2}{\lambda}$ and $y = -\frac{3}{2\lambda}$. Substituting into the third equation,

$$\frac{4}{\lambda^2} + \frac{9}{4\lambda^2} - 25 = 0.$$

Multiplying by λ^2,

$$4 + \frac{9}{4} - 25\lambda^2 = 0.$$

This simplifies to $\lambda^2 = \frac{1}{4}$, or $\lambda = \pm\frac{1}{2}$. This gives us two possible locations for the max and min: $(-4, -3)$ (corresponding to $\lambda = \frac{1}{2}$) and $(4, 3)$ (corresponding to $\lambda = -\frac{1}{2}$).

We're guaranteed that one of these is a max and one a min, so to see which is which we just need to compare the values. We find $f(-4, -3) = -26$ and $f(4, 3) = 24$, so we conclude that $(-4, -3)$ is the minimum and $(4, 3)$ is the maximum. □

Exercise 12. Let R be the region bounded by the curves

$$y = x, \qquad x = \sqrt{y}.$$

Compute

$$\iint_R xy\, dy\, dx.$$

Solution. We could switch the order of integration (i.e. integrate y first) and get the same result, but let's do it in the order given in the problem. Since we're integrating y first we should put our boundary curves in the form of y as a function of x: $y = x$ and $y = x^2$. Drawing a picture we see that x^2 lies below x in the relevant region, and that the curves intersect at $x = 0$ and $x = 1$, so our integral is

$$\begin{aligned}
\int_0^1 \int_{x^2}^x xy\, dy\, dx &= \int_0^1 \left[\frac{xy^2}{2}\right]_{y=x^2}^x dx \\
&= \int_0^1 \frac{x^3}{2} - \frac{x^5}{2} dx \\
&= \left[\frac{x^4}{8} - \frac{x^6}{12}\right]_{x=0}^1 \\
&= \frac{1}{24}.
\end{aligned}$$

□

Exercise 13. Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ for

$$f(x, y, z) = x^2 y + 3z + xe^{y^2 z}.$$

Solution. Using the power rule and rules for exponential functions, we find

$$\frac{\partial f}{\partial x} = 2xy + e^{y^2 z}, \qquad \frac{\partial f}{\partial y} = x^2 + 2xyz\, e^{y^2 z}, \qquad \frac{\partial f}{\partial z} = 3 + xy^2 e^{y^2 z}.$$

□

Exercise 14. If three positive numbers sum to 9, what is the largest their product can be?

Solution. Translating this into equations, we're told that $x + y + z = 9$ and we want to maximize xyz. This is constrained optimization, so we'll use Lagrange Multipliers. Define

$$F(x, y, z, \lambda) = xyz + \lambda(x + y + z - 9).$$

Then

$$\frac{\partial F}{\partial x} = yz + \lambda = 0, \qquad \frac{\partial F}{\partial y} = xz + \lambda = 0, \qquad \frac{\partial F}{\partial z} = xy + \lambda = 0, \qquad \frac{\partial F}{\partial \lambda} = x + y + z - 9 = 0.$$

We're also told that x, y, z are positive, in particular not zero, so we can divide by them. The first three equations tell us

$$-\lambda = yz, \qquad -\lambda = xz, \qquad -\lambda = xy.$$

Thus $yz = xz$, so dividing by z we see $x = y$. Similarly we find $y = z$, so in fact all three of x, y, z are equal. Now the last equation $x + y + z - 9 = 0$ tells us that $x = y = z = 3$, so the maximum is achieved at the point $(3, 3, 3)$, and the maximum value is $3 \cdot 3 \cdot 3 = 27$. □