Final Exam Review

MATH 16B Spring 2016

Exercise 1. Find both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions.

1. $f(x, y) = \frac{\sin(xy)}{x^2}$
2. $f(x, y) = ye^{xy}$
3. $f(x, y) = x^y$

Exercise 2. Find all critical points of $f(x, y) = 2x^2 + y^3 - x - 12y + 7$, and label each as a maximum, minimum, or neither.

Exercise 3. Find the values of x, y, z that maximize $3x + 5y + z - x^2 - y^2 - z^2$ subject to the constraint $6 - x - y - z = 0$.

Exercise 4. Let R be the region bounded by the x-axis, the line $x = 2$, and the graph of $y = x^2$. Compute the following double integral.

$$\iint_R x^2 + y \, dy \, dx$$

Exercise 5. Compute the following indefinite integrals.

1. $\int \sin x \cos x \, dx$
2. $\int \frac{\ln x}{x^3} dx$

Exercise 6. Compute the following integral.

$$\int_0^\infty xe^{-x^2} dx.$$

Exercise 7. Solve the following initial value problems.

1. $y' = y^2 \sin t$, with $y(\pi/2) = 1$
2. $ty' + y = \ln t$, with $y(e) = 0$

Exercise 8. Compute the third order Taylor polynomial of $\cos x$ at $x = 0$, and use it to estimate $\cos 1$. Use the remainder formula to give an upper bound on the error of this estimate.

Exercise 9. Decide whether each series converges or diverges. If it is a convergent geometric series, find the sum.

1. $\sum_{n=1}^\infty \frac{\sin^2 n}{n^2}$

2. $\sum_{n=0}^{\infty} \frac{3}{5^{n+1}}$

3. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Exercise 10. Compute directly (i.e. by taking derivatives) the Taylor series of $\frac{1}{1-x}$ at $x = 0$. Then use this to compute the Taylor series of $\arctan x$ at $x = 0$. Put your answer in summation notation $\sum a_i x^i$. (Hint: $\arctan x = \int \frac{1}{1+x^2} dx$).

Exercise 11. Consider a continuous random variable with probability density function $f(x) = 3x^2$, $0 \leq x \leq 1$.

1. Verify that this is a probability density function.
2. Compute the probability that the outcome is at most $\frac{1}{2}$, i.e. $P(X \leq \frac{1}{2})$.
3. What is the expected value of this random variable?
4. What is its variance?

Exercise 12. Let X be a normal random variable with mean 1 and standard deviation 3. Find $P(|X| < 1)$.

Exercise 13. Consider the process of rolling a (fair six-sided) die repeatedly until the result is a 6. Let X be a random variable representing the total number of rolls preceeding the first 6 (not including the 6).

1. What is the probability that the total number of rolls is n, i.e. $P(X = n)$?
2. What is the expected total number of rolls?

Exercise 14. Consider the process of rolling a (fair six-sided) die 100 times. Let X be the number of 6s among the 100 rolls.

1. What is the expected value of X?
2. We may assume X to be a Poisson random variable. Under this assumption, what is the probability of no 6s whatsoever in the 100 rolls?
3. Give the probability of the number of 6s being n, i.e. $P(X = n)$.