# Final Exam Review

## MATH 16B Spring 2016

**Exercise 1.** Find both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions.

1. $f(x,y) = \frac{\sin(xy)}{x^2}$
2. $f(x,y) = ye^{xy}$
3. $f(x,y) = x^y$

*Solution.* For the first,

$$\frac{\partial f}{\partial x} = \frac{x^2 y\cos(xy) - 2x\sin(xy)}{x^4} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x^3\cos(xy)}{x^4}.$$

For the second,

$$\frac{\partial f}{\partial x} = y^2 e^{xy} \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{xy} + xye^{xy}.$$

For the third,

$$\frac{\partial f}{\partial x} = yx^{y-1} \quad \text{and} \quad \frac{\partial f}{\partial y} = x^y \ln(x).$$

□

**Exercise 2.** Find all critical points of $f(x,y) = 2x^2 + y^3 - x - 12y + 7$, and label each as a maximum, minimum, or neither.

*Solution.* We find critical points using the first derivative test. The first derivatives are

$$\frac{\partial f}{\partial x} = 4x - 1, \qquad \frac{\partial f}{\partial y} = 3y^2 - 12.$$

The first has solution $x = \frac{1}{4}$, and the second has solutions $y = \pm 2$. Thus our critical points are $(\frac{1}{4}, 2)$ and $(\frac{1}{4}, -2)$.

Now we use the second derivative test to classify these critical points. The second derivatives are

$$\frac{\partial^2 f}{\partial x \partial x} = 4, \qquad \frac{\partial^2 f}{\partial x \partial y} = 0, \qquad \frac{\partial^2 f}{\partial y \partial y} = 6y,$$

and

$$D(x,y) = 24y.$$

At $(\frac{1}{4}, 2)$ we have $D > 0$ and $\frac{\partial^2 f}{\partial x \partial x} > 0$, so $(\frac{1}{4}, 2)$ is a minimum. At $(\frac{1}{4}, -2)$ we have $D < 0$, so $(\frac{1}{4}, -2)$ is neither a maximum nor minimum. □

**Exercise 3.** Find the values of $x, y, z$ that maximize $3x + 5y + z - x^2 - y^2 - z^2$ subject to the constraint $6 - x - y - z = 0$.

*Solution.* For constrained optimization we use Lagrange multipliers. The Lagrange function is

$$F(x, y, z, \lambda) = 3x + 5y + z - x^2 - y^2 - z^2 + \lambda(6 - x - y - z).$$

The first derivatives of this function are

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 3 - 2x - \lambda, \\
\frac{\partial F}{\partial y} &= 5 - 2y - \lambda, \\
\frac{\partial F}{\partial z} &= 1 - 2z - \lambda, \\
\frac{\partial F}{\partial \lambda} &= 6 - x - y - z.
\end{aligned}$$

Solving the first three equations for $x, y, z$ respectively and substituting into the fourth equation, we find $\lambda = -1$. Then substituting this into the first three equations we can solve to find $x = 2$, $y = 3$, and $z = 1$. These are the values that maximize our function subject to the constraint (no need to verify that it is a maximum). □

**Exercise 4.** Let $R$ be the region bounded by the $x$-axis, the line $x = 2$, and the graph of $y = x^2$. Compute the following double integral.

$$\iint_R x^2 + y \, dy \, dx$$

*Solution.* Since the integral is written $dy\, dx$, we integrate with respect to $y$ first and then $x$. The $y$-bounds should thus be from 0 to $x^2$, and the $x$-bounds should be from 0 to 2. The integral is then

$$\begin{aligned}
\iint_R x^2 + y \, dy \, dx &= \int_0^2 \int_0^{x^2} x^2 + y \, dy \, dx \\
&= \int_0^2 \left[ x^2 y + \frac{y^2}{2} \right]_0^{x^2} dx \\
&= \int_0^2 \frac{3}{2} x^4 dx \\
&= \left[ \frac{3x^5}{10} \right]_0^2 \\
&= \frac{48}{5}.
\end{aligned}$$

□

**Exercise 5.** Compute the following indefinite integrals.

1. $\int \sin x \cos x \, dx$
2. $\int \frac{\ln x}{x^3} dx$

*Solution.* The first is substitution; let's use $u = \sin x$. Then $du = \cos x\,dx$, so the integral is

$$\int \sin x \cos x\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{\sin^2(x)}{2} + c.$$

(If you use $u = \cos x$ you'll get a different looking answer, but a trig identity shows they're the same up to a constant).

The second is integration by parts; let's use

$$u = \ln x, \quad du = \frac{1}{x}dx, \quad v = -\frac{1}{2x^2}, \quad dv = \frac{1}{x^3}dx.$$

Then the integral is

$$\begin{aligned}\int \frac{\ln x}{x^3}dx &= -\frac{\ln x}{2x^2} - \int -\frac{1}{2x^3}dx \\ &= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c.\end{aligned}$$

□

**Exercise 6.** Compute the following integral.

$$\int_0^\infty xe^{-x^2}dx.$$

*Solution.* Using the substitution $u = -x^2$ and $du = -2x\,dx$, the integral is

$$\begin{aligned}\int_0^\infty xe^{-x^2}dx &= \lim_{a\to\infty}\int_0^a xe^{-x^2}dx \\ &= \lim_{a\to\infty} -\frac{1}{2}\int_{x=0}^{x=a} e^u du \\ &= \lim_{a\to\infty} -\frac{1}{2}\left[e^u du\right]_{x=0}^{x=a} \\ &= \lim_{a\to\infty} -\frac{1}{2}\left[e^{-x^2}du\right]_0^a \\ &= \lim_{a\to\infty} -\frac{1}{2}e^{-a^2} + \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

□

**Exercise 7.** Solve the following initial value problems.

1. $y' = y^2 \sin t$, with $y(\pi/2) = 1$
2. $ty' + y = \ln t$, with $y(e) = 0$

*Solution.* The first can be solved using separation of variables.

$$\begin{aligned}\frac{dy}{dt} &= y^2 \sin t \\ \int \frac{1}{y^2}dy &= \int \sin t\,dt \\ -\frac{1}{y} &= -\cos(t) + c \\ y &= \frac{1}{\cos(t) + c}\end{aligned}$$

Plugging in the initial condition $y(\pi/2) = 1$ gives $1 = \frac{1}{0+c}$, so we find $c = 1$, and the solution is

$$y = \frac{1}{\cos(t) + 1}.$$

The second can be solved by integrating factors. First, we put it into the standard form for first order linear differential equations.

$$y' + \frac{1}{t}y = \frac{\ln t}{t}$$

So $a(t) = \frac{1}{t}$ and $b(t) = \frac{\ln t}{t}$. The integrating factor is

$$e^{\int a(t)dt} = e^{\int \frac{1}{t}dt} = e^{\ln t} = t.$$

Multiplying our differential equation by the integrating factor $t$, we obtain the new equation

$$ty' + y = \ln t,$$

or

$$\frac{d}{dt}[ty] = \ln t.$$

Now we can integrate to solve the differential equation. Recall that we integrate $\ln t$ by parts, using $u = \ln t$ and $dv = dt$ (so $du = \frac{1}{t}dt$ and $v = t$).

$$\begin{aligned}
ty &= \int \ln t \, dt \\
ty &= t \ln t - \int dt \\
ty &= t \ln t - t + c \\
y &= \ln t - 1 + \frac{c}{t}
\end{aligned}$$

The condition $y(e) = 0$ gives $0 = 1 - 1 + \frac{c}{e}$, so we see $c = 0$, and the solution is

$$y = \ln t - 1.$$

□

**Exercise 8.** Compute the third order Taylor polynomial of $\cos x$ at $x = 0$, and use it to estimate $\cos 1$. Use the remainder formula to give an upper bound on the error of this estimate.

*Solution.* The first three (and zeroth) derivatives of $\cos x$ and their values at $x = 0$ are as follows.

$$\begin{aligned}
f(x) &= \cos x & f(0) &= 1 \\
f'(x) &= -\sin x & f'(0) &= 0 \\
f''(x) &= -\cos x & f''(0) &= -1 \\
f'''(x) &= -\sin x & f'''(0) &= 0
\end{aligned}$$

Thus the third order Taylor polynomial of $\cos x$ at $x = 0$ is

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 - \frac{x^2}{2}.$$

Our estimate for $\cos 1$ is $p_3(1) = 1 - \frac{1}{2} = \frac{1}{2}$.

Now we use the remainder formula to estimate the error. The fourth derivative of $\cos x$ is $\cos x$ again, and to use the remainder formula we need a bound on $|f^{(4)}(x)| = |\cos x|$ for $x$ between 0 and 1. We have

$$|f^{(4)}(x)| = |\cos x| \leq 1$$

for $0 \leq x \leq 1$, because $-1 \leq \cos x \leq 1$ for all $x$. Now applying the remainder formula

$$|\cos 1 - p_3(1)| \leq \frac{M}{(n+1)!}|b-a|^{n+1}$$

with $a = 0$, $b = 1$, $n = 3$, and $M = 1$, we find the error is

$$\left|\cos 1 - \frac{1}{2}\right| \leq \frac{1}{4!}|1-0|^4 = \frac{1}{24}.$$

□

**Exercise 9.** Decide whether each series converges or diverges. If it is a convergent geometric series, find the sum.

1. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$
2. $\sum_{n=0}^{\infty} \frac{3}{5^{n+1}}$
3. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

*Solution.* For the first we use the comparison test. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, because it is a $p$-series with $p > 1$ (or by the integral test). Also $\frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}$, and both series are positive, so the comparison test implies that $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ converges as well.

The second is a geometric series with $a = 3/5$ and $r = 1/5$. Since $|r| < 1$ the series converges, and the sum is $\frac{a}{1-r} = \frac{3/5}{1-1/5} = \frac{3}{4}$.

For the third we use the integral test (although we could also use the comparison test, comparing with $\frac{1}{n^2}$ again). Note that $f(x) = \frac{1}{x(\ln x)^2}$ is positive, continuous, and decreasing. Also (using the substitution $u = \ln x$ and $du = \frac{1}{x}dx$),

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^2}dx &= \lim_{a\to\infty} \int_2^a \frac{1}{x(\ln x)^2}dx \\
&= \lim_{a\to\infty} \int_{x=2}^{x=a} \frac{1}{u^2}du \\
&= \lim_{a\to\infty} \left[-\frac{1}{u}\right]_{x=2}^{x=a} \\
&= \lim_{a\to\infty} \left[-\frac{1}{\ln x}\right]_2^a \\
&= \lim_{a\to\infty} -\frac{1}{\ln a} + \frac{1}{\ln 2} \\
&= \frac{1}{\ln 2},
\end{aligned}$$

so the integral is convergent. Thus the sum $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent as well. □

**Exercise 10.** Compute directly (i.e. by taking derivatives) the Taylor series of $\frac{1}{1-x}$ at $x = 0$. Then use this to compute the Taylor series of $\arctan x$ at $x = 0$. Put your answer in summation notation $\sum a_i x^i$. (Hint: $\arctan x = \int \frac{1}{1+x^2} dx$).

*Solution.* The derivatives of $\frac{1}{1-x}$ and their values at $x = 0$ are as follows.

$$\begin{aligned}
f(x) &= \frac{1}{1-x} & f(0) &= 1 \\
f'(x) &= \frac{1}{(1-x)^2} & f'(0) &= 1 \\
f''(x) &= \frac{2}{(1-x)^3} & f''(0) &= 2 \\
f'''(x) &= \frac{3 \cdot 2}{(1-x)^4} & f'''(0) &= 3 \cdot 2 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \frac{n!}{(1-x)^{n+1}} & f^{(n)}(0) &= n!
\end{aligned}$$

Thus the Taylor series of $\frac{1}{1-x}$ at $x = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n.$$

Now $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$, so we can get the Taylor series of $\frac{1}{1+x^2}$ at $x = 0$ by substituting $-x^2$ for $x$ in the above Taylor series.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Finally, since $\arctan x = \int \frac{1}{1+x^2} dx$, we can get the Taylor series of $\arctan x$ at $x = 0$ by integrating term by term the Taylor series of $\frac{1}{1+x^2}$.

$$\arctan x = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

□

**Exercise 11.** Consider a continuous random variable with probability density function $f(x) = 3x^2$, $0 \leq x \leq 1$.

1. Verify that this is a probability density function.
2. Compute the probability that the outcome is at most $\frac{1}{2}$, i.e. $P(X \leq \frac{1}{2})$.
3. What is the expected value of this random variable?
4. What is its variance?

*Solution.* A probability density function must be positive and integrate to 1. Certainly $3x^2 \geq 0$ for all $x$, and

$$\int_0^1 3x^2 dx = x^3 \Big|_0^1 = 1.$$

Thus $f(x) = 3x^2$, $0 \leq x \leq 1$ is indeed a probability density function.

The probability of an outcome at most $\frac{1}{2}$ is

$$P(X \leq \frac{1}{2}) = \int_0^{1/2} 3x^2 dx = x^3 \Big|_0^{1/2} = \frac{1}{8}.$$

The expected value is

$$\mathrm{E}(X) = \int_0^1 x f(x) dx = \int_0^1 3x^3 dx = \frac{3}{4} x^4 \Big|_0^1 = \frac{3}{4}.$$

The variance is

$$\mathrm{Var}(X) = \int_0^1 x^2 f(x) dx - \mathrm{E}(X)^2 = \int_0^1 3x^4 dx - \frac{9}{16} = \frac{3}{5} x^5 \Big|_0^1 - \frac{9}{16} = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

□

**Exercise 12.** Let $X$ be a normal random variable with mean 1 and standard deviation 3. Find $P(|X| < 1)$.

*Solution.* First of all note $|X| < 1$ is equivalent to $-1 < X < 1$. Now we transform to the standard normal distribution $Z = \frac{X-1}{3}$.

$$P(|X|) = P(-1 < X < 1) = P\left(\frac{-1-1}{3} < \frac{X-1}{3} < \frac{1-1}{3}\right) = P\left(-\frac{2}{3} < Z < 0\right)$$

Now we use the symmetry about 0 of the normal distribution.

$$P\left(-\frac{2}{3} < Z < 0\right) = P\left(0 < Z < \frac{2}{3}\right)$$

This we can finally look up in our table: corresponding to $\frac{2}{3} \approx 0.67$ is the value 0.2486, indicating that

$$P(|X| < 1) = P\left(0 < Z < \frac{2}{3}\right) = 0.2486.$$

□

**Exercise 13.** Consider the process of rolling a (fair six-sided) die repeatedly until the result is a 6. Let $X$ be a random variable representing the total number of rolls preceeding the first 6 (not including the 6).

1. What is the probability that the total number of rolls is $n$, i.e. $P(X = n)$?
2. What is the expected total number of rolls?

*Solution.* This is a geometric random variable, with probability $p = \frac{5}{6}$ of failure and probability $1 - p = \frac{1}{6}$ of success.

Precisely $n$ rolls total means $n$ failures (each with probability $p$) followed by a single success (with probability $1 - p$), and the probability of this is

$$P(X = n) = p^n (1 - p) = \left(\frac{5}{6}\right)^n \frac{1}{6}.$$

The expected value of a geometric random variable is $\frac{p}{1-p}$, which in this case is

$$\mathrm{E}(X) = \frac{5/6}{1-5/6} = \frac{5/6}{1/6} = 5.$$

[Pro tip: if you forget this formula, or just for general cultural enlightenment, you can derive it thus. The expected value is

$$\mathrm{E}(X) = \sum_{n=0}^{\infty} nP(X=n) = \sum_{n=0}^{\infty} np^n(1-p) = p(1-p)\sum_{n=0}^{\infty} np^{n-1}.$$

Now recognize $\sum_{n=0}^{\infty} nx^{n-1}$ as the power series of $\frac{1}{(1-x)^2}$ (the derivative of $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$), so substitute in $\frac{1}{(1-p)^2}$ for the infinite series to find the expected value is $\frac{p}{1-p}$.] □

**Exercise 14.** Consider the process of rolling a (fair six-sided) die 100 times. Let $X$ be the number of 6s among the 100 rolls.

1. What is the expected value of $X$?
2. We may assume $X$ to be a Poisson random variable. Under this assumption, what is the probability of no 6s whatsoever in the 100 rolls?
3. Give the probability of the number of 6s being $n$, i.e. $P(X=n)$.

*Solution.* The probability of a 6 on any individual roll is $\frac{1}{6}$, so the expected number of 6s in 100 rolls is $\frac{100}{6}$.

If $X$ is Poisson then it has parameter $\lambda = \frac{100}{6}$, because $\lambda$ is the expected value. Now the probability of no 6s is $P(X=0) = e^{-\lambda} = e^{-100/6}$. (You do not need to simplify this, but just for the record it comes out to about $6 \times 10^{-8}$. The chances are not good.)

For a Poisson random variable, $P(X=n) = \frac{\lambda^n}{n!}e^{-\lambda}$. □