Worksheet 8 Solutions

MATH 1A Fall 2015

for 24 November 2015

Exercise 8.1. Find the derivatives of the following functions.

1.
$$f(x) = \int_0^x e^{t^2} dt$$

2.
$$g(x) = \int_0^{x^2} \frac{\sin t}{t} dt$$

Solution. Recall that the (first part of the) fundamental theorem of calculus states: for a continuous function f,

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

For the first problem we can apply this immediately to find

$$\frac{d}{dx} f(x) = \frac{d}{dx} \int_0^x e^{t^2} dt = e^{x^2}.$$

For the second problem, first define

$$\tilde{g}(x) = \int_0^x \frac{\sin t}{t} dt.$$

Note $g(x) = \tilde{g}(x^2)$. The point is that we can apply the fundamental theorem of calculus to find the derivative of $\tilde{g}$:

$$\frac{d}{dx} \tilde{g}(x) = \frac{d}{dx} \int_0^x \frac{\sin t}{t} dt = \frac{\sin x}{x},$$

and we can use the chain rule to find the derivative of g in terms of the derivative of $\tilde{g}$:

$$\frac{d}{dx} g(x) = \frac{d}{dx} \tilde{g}(x^2) = \tilde{g}'(x^2) 2x.$$

Combining these, we find

$$\frac{d}{dx} g(x) = \frac{\sin(x^2)}{x^2} 2x = \frac{2\sin(x^2)}{x}.$$

□

Exercise 8.2. Find antiderivatives of the following functions.

1. $\frac{1}{1-x}$
2. $\cos(2x)$.

Solution. It's possible to see these antiderivatives immediately from the function, but let's use a u-substitution to make things perfectly clear.

First of all, we know $\ln|x|$ is an antiderivative of $\frac{1}{x}$ (i.e. $\frac{d}{dx}\ln|x| = \frac{1}{x}$). Setting $u = 1-x$, we have $\frac{du}{dx} = -1$, so $-du = dx$, so

$$\int \frac{1}{1-x}dx = -\int \frac{1}{u}du = -\ln|u| + c = -\ln|1-x| + c.$$

We also know $\sin x$ is an antiderivative of $\cos x$ (i.e. $\frac{d}{dx}\sin x = \cos x$). Setting $u = 2x$, we have $\frac{du}{dx} = 2$, so $\frac{1}{2}du = dx$, so

$$\int \cos(2x)dx = \frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(u) + c = \frac{1}{2}\sin(2x) + c.$$

□