Worksheet 7 Solutions

MATH 1A Fall 2015

for 3 November 2015

Exercise 7.1. Minimize the surface area of a cylindrical tube (including top and bottom) with volume 3. (Find the value of the minimal surface area and the dimensions that make it so.)

Solution. Let r be the radius and h the height of our cylinder. The surface area, which we want to minimize, is

$$2\pi r^2 + 2\pi rh$$

and the volume, which is a constraint, is

$$3 = \pi r^2 h.$$

Solving the volume constraint for h gives

$$h = \frac{3}{\pi r^2},$$

and substituting this into the surface area equation we obtain

$$2\pi r^2 + \frac{6}{r}.$$

This is the function we want to maximize. Its derivative is

$$4\pi r - \frac{6}{r^2},$$

which is equal to zero only when $r = \sqrt[3]{3/2\pi}$ and never undefined (except at $r = 0$, but we ignore this because it has no physical meaning). Thus our only critical point is $r = \sqrt[3]{3/2\pi}$.

The second derivative of surface area is

$$4\pi + \frac{12}{r^3},$$

and at $r = \sqrt[3]{3/2\pi}$ this comes out to

$$4\pi + \frac{12}{3/2\pi}.$$

This is positive, so the second derivative test tells us that this critical point is a minimum. Thus the minimum surface area is achieved with

$$r = \sqrt[3]{3/2\pi}, \qquad h = \frac{3}{\pi(3/2\pi)^{2/3}}$$

and this minimal surface area is

$$2\pi(3/2\pi)^{2/3} + \frac{6}{\sqrt[3]{3/2\pi}}$$

□

Exercise 7.2. Find the point(s) on the parabola $y = x^2$ closest to the point $(-12, 15/2)$.

Proof. We want to minimize the distance

$$\sqrt{(x+12)^2 + (y-15/2)^2}$$

from a point (x, y) to $(-12, 15/2)$ subject to the constraint $y = x^2$. Substituting this constraint into our equation, we want to minimize

$$\sqrt{(x+12)^2 + (x^2-15/2)^2}.$$

The derivative of this function is

$$\frac{1}{2\sqrt{(x+12)^2 + (x^2-15/2)^2}}(2(x+12) + 2(x^2-15/2)2x).$$

This is never undefined, since the expression inside the square root is never negative and the denominator is never zero. Since the left factor is never zero, the derivative is equal to zero precisely when

$$\begin{aligned} 2(x+12) + 2(x^2-15/2)2x &= 0 \\ 4x^3 - 28x + 24 &= 0 \\ 4(x-1)(x-2)(x+3) &= 0 \end{aligned}$$

so our critical points are $x = 1, 2, -3$. (If the factorization seems mysterious, don't worry about it too much; if you really want to see how you might factor such a thing, look up the "rational roots test").

To determine the nature of these critical points we can use the first derivative test. Rewriting the derivative as

$$(\text{something positive})(x-1)(x-2)(x+3)$$

we can see that the first derivative is negative on $(-\infty, -3)$; positive on $(-3, 1)$; negative on $(1, 2)$; and positive on $(2, \infty)$. Thus $2, -3$ are local minima and 1 is a local maximum.

Note that since the derivative is negative on $(-\infty, -3)$ and positive on $(2, \infty)$, one of $2, -3$ is an absolute minimum. To see which it is we simply plug in to our original distance equation. At $x = 2$ the distance is

$$\sqrt{14^2 + (-7/2)^2}$$

and at $x = -3$ the distance is

$$\sqrt{9^2 + (3/2)^2}.$$

The distance at $x = -3$ is smaller, so this is the absolute minimum, and we conclude that the closest point to $(-12, 15/2)$ on the parabola $y = x^2$ is the point $(-3, 9)$. □