Worksheet 3 Solutions

MATH 1A Fall 2015

for 22 September 2015

The squeeze theorem is a useful tool for evaluating limits, and to use it properly it probably helps to understand what it is. It's also useful to be able to translate back and forth between intuitive ways of saying things and precise ways of saying things (and this example in particular will be necessary when we're proving the squeeze theorem or applying it to prove limits).

Exercise 3.1. State the squeeze theorem. Next, if your statement includes (anything like) the phrase "when x is near a, except possibly at a, we have $f(x) \leq g(x) \leq h(x)$", replace this intuitive phrase with a precise mathematical statement.

Solution. The squeeze theorem states: Let f, g, h be real-valued functions, and let $a, L \in \mathbb{R}$. Suppose that when x is near a, except possibly at a, we have $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x).$$

Then $\lim_{x \to a} g(x) = L$.

To be more precise we can replace "when x is near a, except possibly at a, we have $f(x) \leq g(x) \leq h(x)$" with "there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) \leq g(x) \leq h(x)$". □

Exercise 3.2. Prove that

$$\lim_{x \to 0} x^2 e^{\sin x} = 0.$$

(Recall e is just some real number, about 2.7ish).

Proof. Note that

$$\begin{aligned} -1 \leq \sin x &\leq 1, && \text{so} \\ e^{-1} \leq e^{\sin x} &\leq e^1 && \text{and} \\ x^2 e^{-1} \leq x^2 e^{\sin x} &\leq x^2 e. \end{aligned}$$

Note also that $\lim_{x \to 0} x^2 e^{-1} = 0 = \lim_{x \to 0} x^2 e$. By the squeeze theorem, we conclude that $\lim_{x \to 0} x^2 e^{\sin x} = 0$. □

We'll probably also need to be able to use all sorts of different limits, so here is one to exercise a different definition of limit than the usual one.

Exercise 3.3. For a function $f(x)$, define what it means to say

$$\lim_{x \to a^+} f(x) = \infty.$$

Then prove that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Proof. We say that $\lim_{x \to a^+} f(x) = \infty$ if for every $N \in \mathbb{R}$ there exists a $\delta > 0$ such that if $a < x < a + \delta$ then $f(x) > N$.

Proof 1: Let $N \in \mathbb{R}$, and set $\delta = \frac{1}{|N|+1}$. Suppose $0 < x < \delta = \frac{1}{|N|+1}$. Then

$$\begin{aligned} x &< \frac{1}{|N|+1} \\ \therefore x(|N|+1) &< 1 \\ \therefore |N|+1 &< \frac{1}{x} \\ \therefore |N| &< \frac{1}{x} \\ \therefore N &< \frac{1}{x}. \end{aligned}$$

Thus for every $N \in \mathbb{R}$ there exists a $\delta > 0$, namely $\delta = \frac{1}{|N|+1}$, such that if $0 < x < \delta$ then $\frac{1}{x} > N$. This proves $\lim_{0^+} \frac{1}{x} = \infty$.

Proof 2: Note that if $0 < x < \delta$ implies $\frac{1}{x} > N$, then for any $N' < N$, we also have $0 < x < \delta$ implies $\frac{1}{x} > N'$. Thus if we prove there exists such a δ for all $N > 0$, then it follows that there exists such a δ for all $N \in \mathbb{R}$.

Let $N > 0$, and set $\delta = \frac{1}{N}$. If $0 < x < \delta = \frac{1}{N}$, then $xN < 1$ and $N < \frac{1}{x}$. This proves $\lim_{x \to 0^+} \frac{1}{x} = \infty$. $\square$