Worksheet 2 Solutions

MATH 1A Fall 2015

for 15 September 2015

Exercise 2.1. Suppose $\lim_{x\to a} f(x) = L$, and let $c \in \mathbb{R}$. Prove that

$$\lim_{x\to a} cf(x) = cL.$$

Proof. First note that if $c = 0$, then $\lim_{x\to a} cf(x) = \lim_{x\to a} 0 = 0 = cL$. This proves the claim if $c = 0$.

Now suppose $c \neq 0$. Let $\varepsilon > 0$. Because $\lim_{x\to a} f(x) = L$, the definition of a limit tells us that there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \frac{\varepsilon}{|c|}$. Choose such a δ.

Now if $0 < |x-a| < \delta$, we have

$$\begin{aligned} & |f(x) - L| < \frac{\varepsilon}{|c|} \\ \text{so} \quad & |c|\,|f(x) - L| < \varepsilon \\ \text{and} \quad & |cf(x) - cL| < \varepsilon. \end{aligned}$$

Thus $\lim_{xtoa} cf(x) = cL$. □

Exercise 2.2. Evaluate, with proof, the limit

$$\lim_{x\to 2} \frac{x^2 + 3x - 10}{x - 2}.$$

[Hint: remember that the limit doesn't care what happens *at* $x = 2$, just what happens *near* $x = 2$. So we can assume $x \neq 2$, and the expression simplifies. After that it should be more familiar.]

Proof. The definition of a limit does not depend on the behavior at $x = 2$, so in the process of evaluating the limit we may assume $x \neq 2$. Supposing $x \neq 2$, we have

$$\frac{x^2 + 3x - 10}{x - 2} = \frac{(x-2)(x+5)}{x-2} = x + 5,$$

and since this is true for all $x \neq 2$ we have

$$\lim_{x\to 2} \frac{x^2 + 3x - 10}{x - 2} = \lim_{x\to 2} x + 5 = 7.$$

(To prove the second equality, i.e. $\lim_{x\to 2} x + 5 = 7$, we can use the usual $\varepsilon - \delta$ style proof, or alternatively we can recall that $x + 5$ is continuous so that plugging in the value gives the limit). □

Exercise 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that f and g are continuous (i.e. continuous at a for every $a \in \mathbb{R}$). Prove that $f + g$ is continuous. In case it's not familiar, $f + g$ is the function that takes a real number x to the value $f(x) + g(x)$. [Hint: I'm not sure how hard this looks, but the proof is *very short*. Know your definitions and limit laws.]

Proof. By definition, f and g being continuous means that

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a)$$

for all $a \in \mathbb{R}$. By the sum law for limits, we conclude that

$$\lim_{x \to a} f(x) + g(x) = f(a) + g(a)$$

for all $a \in \mathbb{R}$, and this is simply the statement that $f + g$ is continuous. □