Worksheet 1 Solutions

MATH 1A Fall 2015

for 8 September 2015

Exercise 0.1. Write down the truth table for the statement "If P then not Q", or equivalently "P implies not Q".

Solution.

		P	
		T	F
Q	T	F	T
	F	T	T

Remember that the only way an "implies" statement is False is when it has the form "T implies F". The other possibilities, "T implies T", "F implies T", "F implies F" are all True. $\square$

It's probably more important to understand how limits work than to memorize the definition, but hopefully if you understand limits well enough then you can come up with the definition.

Exercise 0.2. State the definition of a limit: Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a, L \in \mathbb{R}$. Then

$$\lim_{x \to a} f(x) = L$$

if ... .

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $a, L \in \mathbb{R}$. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Intuitively: $\lim_{x\to a} f(x) = L$ if, as x get closer and closer to a, the values $f(x)$ get closer and closer to L. The variable ε corresponds to $f(x)$ getting closer and closer to L, and the variable δ corresponds to x getting closer and closer to a. The definition again:

for every $\varepsilon > 0$ (however close you want $f(x)$ to be to L) there exists a $\delta > 0$ (we can get x close enough to a) such that if $0 < |x - a| < \delta$ (such that if x is close enough to a) then $|f(x) - L| < \varepsilon$ (then $f(x)$ is close enough to L). $\square$

Here's a limit problem for a quadratic polynomial. See if you can remember the trick with δ we used to solve it.

Exercise 0.3. Prove that

$$\lim_{x\to -1} x^2 - 2x + 2 = 5.$$

Proof. Let $\varepsilon > 0$, and set $\delta = \min(1, \varepsilon/5)$.

Suppose $0 < |x + 1| < \delta$ (note $|x + 1| = |x - (-1)|$). Then $|x + 1| < \varepsilon/5$. Also $|x + 1| < 1$, which is the same as $-1 < x + 1 < 1$; subtracting 4 from each term we get $-5 < x - 3 < -3$, and this implies $|x - 3| < 5$.

Now

$$\begin{aligned}
& |x^2 - 2x + 2 - 5| & \\
&= |x^2 - 2x - 3| & \\
&= |(x - 3)(x + 1)| & \\
&= |x - 3||x + 1| & \\
&< 5|x + 1| & \text{because } |x - 3| < 5 \\
&< 5 \cdot \varepsilon/5 & \text{because } |x + 1| < \varepsilon/5 \\
&= \varepsilon. &
\end{aligned}$$

This shows $|x^2 - 2x + 2 - 5| < \varepsilon$.

For every $\varepsilon > 0$ we've produced a $\delta > 0$ such that if $0 < |x + 1| < \delta$ then $|x^2 - 2x + 2 - 5| < \varepsilon$. Thus $\lim_{x\to -1} x^2 - 2x + 2 = 5$. $\square$

So that's just the proof; here's some scratch work and explanation. We want to choose δ so that when x is near -1 (where "near" means $0 < |x + 1| < \delta$) then $|x^2 - 2x + 2 - 5| = |x - 3||x + 1|$ is smaller than ε. How can we choose δ to accomplish this?

The part of the equation we control is $|x+1|$, which we can make as small as we like (by choosing a small δ). The other part, $|x - 3|$, we *can't* make as small as we want, because x is close to -1, which means x isn't particularly close to 3. However, as long as the other part $|x - 3|$ isn't getting too big, we can make $|x + 1|$ as small as we like to get the whole thing $|x - 3||x + 1|$ as small as we like (which is to say, smaller than ε).

So our strategy is to make $|x-3|$ smaller than some constant, say C, and then to make $|x+1|$ smaller than ε/C. This is where the two δs come in: the first, say δ_1, is used to make $|x-3| < C$; the other, say δ_2, is used to make $|x+1| < \varepsilon/C$. (Note that for the second we can just take $\delta_2 = \varepsilon/C$; once we know what C is, at any rate).

We can set δ_1 to be anything. To illustrate this, let's not pick it at all yet. Remember that choosing δ_1 gives us $0 < |x+1| < \delta_1$, and we want to show $|x-3| < C$ (where again C is just some number, which we haven't set yet). To go from $|x+1| < \delta_1$ to $|x-3| < C$ we do the following thing:

$$\begin{array}{ll}
|x+1| < \delta_1 & \\
-\delta_1 < x+1 < \delta_1 & \text{rewrite the absolute value as inequalities} \\
-\delta_1 - 4 < x-3 < \delta_1 - 4 & \text{change } x+1 \text{ to } x-3 \text{ by subtracting } 4 \\
|x-3| < |-\delta_1 - 4| = \delta_1 + 4 & \text{note that since } \delta_1 \text{ is positive, } |-\delta_1 - 4| > |\delta_1 - 4|.
\end{array}$$

We wanted $|x-3| < C$, and in this case we find that C is $\delta_1 + 4$. So, for example, if we choose $\delta_1 = 1$ then we get $|x-3| < 1+4 = 5$.

Since we can choose δ_1 to be anything we want, we choose this first, and once we've done that we find C and set $\delta_2 = \varepsilon/C$. So maybe we choose $\delta_1 = 1$, and find $C = 5$, so we then choose $\delta_2 = \varepsilon/5$.

Now as always when we have two δs for two different jobs, we choose $\delta = \min(\delta_1, \delta_2)$. This is because we never lose anything by making δ smaller; if δ_1 is smaller than δ_2, then δ_1 does its job and it also does δ_2's job.

After choosing δ, the rest of the proof is pretty much the same as the scratch work.