Quiz 6 Solutions

MATH 1A Fall 2015

22 October 2015

Exercise 6.1. The isotope Carbon-15 is radioactive with a half-life of about 2 seconds. We'll measure radioactivity in terms of decays per second, i.e. the number of atoms per second lost to radioactive decay.

Suppose we are given (say at $t = 0$) a sample of Carbon-15 with 10^{23} atoms, and assume that our sample decays exponentially.

1. How many decays per second will we measure at $t = 0$?
2. Suppose in this scenario that 10^{10} decays per second is the maximum safe level of radiation. At what time will our sample become safe to handle?
3. At that time, how much Carbon-15 is left?
4. How long will it take for our sample to disappear completely? (I.e. how long until there is less than 1 atom left?)

Solution. Let's first write an exponential model $y = ce^{kt}$ for $y =$ # atoms in terms of $t =$ seconds.

The constant c as usual is the initial value $y(0)$, so $c = 10^{23}$. We can get a second data point using the half-life: after two seconds half of our sample will have decayed, so $y(2) = 10^{23}/2$. Plugging this into our equation,

$$\frac{1}{2}10^{23} = 10^{32}e^{2k},$$

and solving for k we find $k = \frac{1}{2}\ln\frac{1}{2}$. Thus our model is

$$y = 10^{23}e^{\frac{1}{2}\ln(\frac{1}{2})t}, \qquad \text{or alternatively} \qquad y = 10^{23}\left(\tfrac{1}{2}\right)^{t/2}.$$

1. The number of decays per second is $-\frac{dy}{dx}$, so this question is asking for $-\frac{dy}{dx}$ at $t = 0$. Using the chain rule and the rules for differentiating exponentials, we find

$$\frac{dy}{dt} = \tfrac{1}{2}\ln\left(\tfrac{1}{2}\right)10^{23}\left(\tfrac{1}{2}\right)^{t/2},$$

and plugging in $t = 0$, we find the number of decays per second to be $-\frac{1}{2}\ln\left(\frac{1}{2}\right)10^{23}$, or about 3.5×10^{22} decays per second.

2. This part asks at what t we will find $\frac{dy}{dx} = -10^{10}$. We just plug this into the derivative formula

$$-10^{10} = \tfrac{1}{2}\ln\left(\tfrac{1}{2}\right)10^{23}\left(\tfrac{1}{2}\right)^{t/2},$$

and solve for t to find

$$t = 2\log_{1/2}\left(-\frac{2\cdot 10^{-13}}{\ln\frac{1}{2}}\right),$$

or about 83 seconds.

3. This question asks for y at the time we've just computed. To find this we plug our time into the original exponential equation, and find

$$y = 10^{23}\left(\frac{1}{2}\right)^{2\log_{1/2}\left(-\frac{2\cdot 10^{-13}}{\ln\frac{1}{2}}\right)/2}$$

which simplifies to

$$y = \frac{2\cdot 10^{10}}{\ln\frac{1}{2}}.$$

4. This part asks at what t we will find $y = 1$, so we plug this condition into our original exponential equation to find

$$1 = 10^{23}\left(\frac{1}{2}\right)^{t/2},$$

and solving for t,

$$t = 2\log_{1/2}(10^{-23})$$

or about 153 seconds. □