Practice Final Exam Solutions

MATH 1A Fall 2015

Problem 1. A 13 foot ladder rests against a wall. The base of the ladder is pushed toward the wall at 2 feet per second. How fast is the top of the ladder moving up the wall when the base is 5 feet from the wall?

Solution. The distance x of the base of the ladder from the wall and the height y of the top of the ladder up the wall (both functions of time t) are related by

$$x^2 + y^2 = 13.$$

We want to find $\frac{dy}{dt}$ when $x = 5$ (and $\frac{dx}{dt} = -2$). Taking an implicit derivative with respect to t, we find

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

From the first equation, plugging in $x = 5$, we find $y = 12$; we also know $\frac{dx}{dt} = -2$, so

$$2(5)(-2) + 2(12)\frac{dy}{dt} = 0,$$

and we find $\frac{dy}{dt} = \frac{5}{6}$. □

Problem 2. Prove that there is a real number x for which $\ln x = \frac{1}{x}$.

Proof. Consider the function $f(x) = \ln x - \frac{1}{x}$. It is continuous on $(0, \infty)$. Furthermore,

$$f(1) = 0 - \frac{1}{1} < 0$$

and

$$f(e) = 1 - \frac{1}{e} > 0.$$

By the intermediate value theorem we conclude that $f(x) = 0$ for some $x \in (1, e)$, and this is a solution to $\ln x = \frac{1}{x}$. □

Problem 3. Find the derivatives of the following functions.

(a) $x^2 e^x$

(b) $\ln(\sec x + \tan x)$

(c) x^x

Solution. (a) By the product rule, $\frac{d}{dx} x^2 e^x = 2xe^x + x^2 e^x$.

(b) Using the quotient rule, we find $\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \sec x \tan x$ and $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \sec^2 x$. Using these together with the chain rule, we find

$$\frac{d}{dx}\ln(\sec x + \tan x) = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} = \sec x.$$

(c) Writing $x^x = e^{x \ln x}$, we can use the chain rule (and product rule) to find

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{x \ln x} = e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x).$$

□

Problem 4. (a) Define what it means to say $\lim_{x \to a^+} f(x) = \infty$.

(b) Prove, using the definition from the previous part, that $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$.

Proof. Suppose for all $M \in \mathbb{R}$ there exists a $\delta > 0$ such that if $a < x < a + \delta$ then $f(x) > M$. Then we say $\lim_{x \to a^+} f(x) = \infty$.

Let $M \in \mathbb{R}$, and assume (without loss of generality) that $M > 0$. Set $\delta = \frac{1}{M}$. Suppose $2 < x < 2 + \delta$, i.e. $0 < x - 2 < \frac{1}{M}$. Then dividing by $x - 2$ and multiplying by M (note both are positive so our inequalities are preserved), we find $M < \frac{1}{x-2}$. Thus $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$. □

Problem 5. State the extreme value theorem.

Solution. If f is a continuous function on a closed interval $[a, b]$, then f has an absolute maximum and absolute minimum on $[a, b]$. □

Problem 6. (a) State the limit definition of the derivative.

(b) Prove, using the definition from the previous part, that $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

Proof. The derivative of a function f at a point x, is the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(if this limit exists).

Using this definition, to show $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ is to show

$$\lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

Starting from the left side, we can simply write

$$\begin{aligned}\lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} &= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\end{aligned}$$

where the last euqality is by the sum law for limits. □

Problem 7. Let f be a differentiable function. Suppose that $f(0) = 0$ and $f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all $x > 0$.

Proof. Let $x > 0$. We know f is differentiable, so by the mean value theorem there is a $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$. Since x and $f'(c)$ are both positive, it must be that $f(x)$ is positive as well. □

Problem 8. State and prove the squeeze theorem.

Solution. Let f, g, h be real-valued functions and $a \in \mathbb{R}$. Suppose that when x is near a, except possibly at a, we have $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x).$$

Then

$$\lim_{x \to a} g(x) = L.$$

The proof is as follows. Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $f(x) \leq g(x) \leq h(x)$.

Since $\lim_{x \to a} f(x) = L$, we can choose $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|f(x) - L| < \varepsilon$. Similarly, since $\lim_{x \to a} h(x) = L$, we can choose $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$ then $|h(x) - L| < \varepsilon$.

Set $\delta = \min(\delta_1, \delta_2, \delta_3)$, and suppose $0 < |x - a| < \delta$.

Then $0 < |x - a| < \delta_1$, so $f(x) \leq g(x) \leq h(x)$.

Also $0 < |x - a| < \delta_2$, so $|f(x) - L| < \varepsilon$, i.e. $L - \varepsilon < f(x) < L + \varepsilon$.

Also $0 < |x - a| < \delta_3$, so $|h(x) - L| < \varepsilon$, i.e. $L - \varepsilon < h(x) < L + \varepsilon$.

Combining these, we see

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon,$$

so $|g(x) - L| < \varepsilon$. □

Problem 9. Find $\frac{dy}{dx}$ if $y^2 x + \ln y = \sin(2x)$.

Solution. Taking an implicit derivative with respect to x,

$$2yx\frac{dy}{dx} + y^2 + \frac{1}{y}\frac{dy}{dx} = 2\cos(2x).$$

Now we simply solve for $\frac{dy}{dx}$, and find

$$\frac{dy}{dx} = \frac{2\cos(2x) - y^2}{2xy + 1/y}.$$

□

Problem 10. Moore's law is the observation that the number of transistors in computer processors has doubled every two years. Suppose a 2011 proccessor has 2.6 billion transistors.

(a) Write a model for the number of transistors in a processor as a function of time.

(b) How many transistors did 1971 processors have?

Proof. (a) Let T be the number of transistors in a processor, and t be time in years since 2011 (i.e. set $t = 0$ to be 2011). Since T doubles every two years the growth is exponential, so we're looking for a model of the form $T(t) = Ce^{kt}$. By plugging in $t = 0$ we find $C = T(0) = 2.6 \times 10^9$. We can find another data point using the fact that T doubles every two years: since $T(0) = 2.6 \times 10^9$, it must be that $T(2) = 5.2 \times 10^9$. Plugging this in to our model, we find

$$5.2 \times 10^9 = 2.6 \times 10^9 \, e^{2t},$$

and solving for k gives $k = \frac{\ln 2}{2}$. Thus we arrive at the model

$$T(t) = 2.6 \times 10^9 \, e^{t \frac{\ln 2}{2}} = 2.6 \times 10^9 \, 2^{t/2}.$$

(b) The year 1971 corresponds to $t = -40$, so the number of transistors in a 1971 processor is

$$T(-40) = 2.6 \times 10^9 \, 2^{-20}.$$

□

Problem 11. If two numbers add up to 6, what is the largest their product can be?

Solution. Let's call our two numbers x and y. We're told that $x + y = 6$, and we want to maximize xy. The first equation can be rearranged as $y = 6 - x$, and substituting this into the second equation we get $x(6 - x) = 6x - x^2$.

To maximize this function, we compute the derivative $\frac{d}{dx} 6x - x^2 = 6 - 2x$. This is never undefined, and it is zero when $x = 3$, so $x = 3$ is our only critical point. The second derivative $\frac{d}{dx} 6 - 2x = -2$ is negative at this point, so by the second derivative test it is indeed a maximum. Thus the maximum is achieved at $x = 3$, $y = 3$ and the maximum product is 9. □

Problem 12. State the fundamental theorem of calculus.

Solution. Part 1: If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t) dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and $\frac{d}{dx} g(x) = f(x)$.

Alternatively: If f is continuous, then

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

Part 2: If F is differentiable, then

$$\int_a^b \frac{d}{dx} F(x) dx = F(b) - F(a).$$

□

Problem 13. Find the antiderivatives of the following functions.

(a) $(x + 2)(x + 4)$

(b) $\tan x$

(c) $x 3^{x^2 + 3}$

Solution. (a) Using the power rule:

$$\int (x+2)(x+4)dx = \int x^2 + 6x + 8dx = \frac{x^3}{3} + 3x^2 + 8x + C.$$

(b) Using u-substitution with $u = \cos x$:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C.$$

(c) Using u-substitution with $u = x^2 + 3$ (note $u = 3^{x^2+3}$ also works):

$$\int x 3^{x^2+3} dx = \frac{1}{2} \int 3^u du = \frac{1}{2} \frac{3^u}{\ln 3} + C = \frac{3^{x^2+3}}{2 \ln 3} + C.$$

□

Problem 14. Evaluate the following limits. Show work, but there is no need to justify each step.

(a) $\lim_{x\to\infty} \frac{(x-1)(2x+2)}{x^2+4x+3}$

(b) $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$

(c) $\lim_{x\to 0} \frac{\sin 2x}{x}$

Solution. (a) By comparing the leading terms of the numerator and denominator,

$$\lim_{x\to\infty} \frac{(x-1)(2x+2)}{x^2+4x+3} = \lim_{x\to\infty} \frac{2x^2-2}{x^2+4x+3} = 2.$$

(We could also be more precise and use l'Hôpital's rule).

(b) Note that $-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$. Also $-x^2$ and x^2 both have a limit of 0 at $x = 0$, so by the squeeze theorem

$$\lim_{x\to 0} x^2 \sin(\tfrac{1}{x}) = 0.$$

(c) By l'Hôpital's rule,

$$\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{2\cos 2x}{1} = 2.$$

□

Problem 15. (a) Define what it means to say a function $f(x)$ is continuous at a point a.

(b) Prove, using the definition above, that $f(x) = 3x + 2$ is continuous at 1.

Proof. A function $f(x)$ is continuous at a if

$$\lim_{x\to a} f(x) = f(a).$$

We want to show that

$$\lim_{x\to 1} 3x + 1 = 5.$$

Let $\varepsilon > 0$, and set $\delta = \varepsilon/3$. Suppose $0 < |x-1| < \delta = \varepsilon/3$. Then

$$|3x + 2 - 5| = |3x - 3| = 3|x-1| < 3\varepsilon/3 = \varepsilon.$$

Thus $\lim_{x\to 1} 3x + 2 = 5$, so $3x + 2$ is continuous at 1. □

Problem 16. Find the 50^{th} derivative of $f(x) = e^{2x+1}$.

Solution. Using the chain rule we see the first derivative is $2e^{2x+1}$, the second is 2^2e^{2x+1}, the third is 2^3e^{2x+1}, and so on. The effect of taking each derivative is to multiply the function by 2, so the 50^{th} derivative is $2^{50}e^{2x+1}$. □

Problem 17. Let P and Q be logical statements, and suppose P **is true** and Q **is false**. Decide whether or not the following statements are true or false.

(a) P and not Q

(b) Q implies P

(c) (not P) if and only if Q

Solution. (a) An "and" statement is true precisely when both of its inputs are true. In our case P is true, and not Q is true (since Q is false), so "P and not Q" is true.

(b) An "implies" statement is only false when the first input is true and the second input is false. In this case the first input Q is false, so "Q implies P" is true.

(c) An "if and only if" statement is true precisely when its inputs are both true or both false. In this case the are both false, so "(not P) if and only if Q" is true.

□