Past Midterm 2 Solutions

6 November 2015

Exercise 0.1. 1. State Rolle's Theorem.

2. State the Mean Value Theorem.

3. Prove the Mean Value Theorem. You may assume Rolle's Theorem.

Solution. 1. Rolle's Theorem states: If a function f is continuous on $[a,b]$ and differentiable on (a,b) and $f(a) = f(b)$ then there is a $c \in (a,b)$ such that $f'(c) = 0$.

2. The Mean Value Theorem states: If a function f is continuous on $[a,b]$ and differentiable on (a,b) then there is a $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

3. Let f be continuous on $[a,b]$ and differentiable on (a,b). Define $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}x$. Then g is continuous on $[a,b]$ because it is a sum of continuous functions, and it is differentiable on (a,b) because it is a sum of differentiable functions. Furthermore

$$\begin{aligned}
g(a) - g(b) &= f(a) - \frac{f(b)-f(a)}{b-a}a - \left(f(b) - \frac{f(b)-f(a)}{b-a}b\right) \\
&= f(a) - f(b) - \frac{f(b)-f(a)}{b-a}a + \frac{f(b)-f(a)}{b-a}b \\
&= f(a) - f(b) + \frac{f(b)-f(a)}{b-a}(b-a) \\
&= f(a) - f(b) + f(b) - f(a) \\
&= 0,
\end{aligned}$$

so $g(a) = g(b)$. Thus we may apply Rolle's Theorem, and conclude that there is a $c \in (a,b)$ such that $g'(c) = 0$. Taking the derivative of the expression defining $g(x)$, we see that

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a},$$

so

$$f'(c) = g'(c) + \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$$

as desired. □

Exercise 0.2. In each of the following cases, evaluate $\frac{dy}{dx}$.

Solution. 1. $y = \sin x \ln|x|$

By the product rule $\frac{dy}{dx} = \cos x \ln|x| + \frac{1}{x}\sin x$.

2. $y = \frac{x^2}{\cos x}$

By the quotient rule $\frac{dy}{dx} = \frac{2x\cos x + x^2 \sin x}{\cos^2 x}$.

3. $y = (10x^4 - 5)^{20}$

   By the chain rule and power rule $\frac{dy}{dx} = 20(10x^4 - 5)^{19} 40x^3$.

4. $y = \sin(x \sin(x))$.

   By the chain rule and product rule $\frac{dy}{dx} = \cos(x \sin(x))(\sin x + x \cos x)$.

5. $y = \tan(x^2)e^{x^3}$

   By the product rule and chain rule $\frac{dy}{dx} = 2x \sec^2(x^2)e^{x^3} + \tan(x^2)3x^2e^{x^3}$.

6. $y^4x - 3y = e^{2x}$ (You should leave your answer in terms of x and y)

   By implicit differentiation

$$\begin{aligned} 4y^3x\frac{dy}{dx} + y^4 - 3\frac{dy}{dx} &= 2e^{2x}, \\ 4y^3x\frac{dy}{dx} - 3\frac{dy}{dx} &= 2e^{2x} - y^4, \\ (4y^3x - 3)\frac{dy}{dx} &= 2e^{2x} - y^4, \\ \frac{dy}{dx} &= \frac{2e^{2x} - y^4}{4y^3x - 3}. \end{aligned}$$

□

Exercise 0.3. Calculate $\frac{d}{dx}x^{e^x}$.

Solution. Rewriting the base x as $e^{\ln x}$, we have

$$\begin{aligned} \frac{d}{dx}x^{e^x} &= \frac{d}{dx}(e^{\ln x})^{e^x} \\ &= \frac{d}{dx}e^{\ln(x)e^x} \\ &= e^{\ln(x)e^x}\frac{d}{dx}\ln(x)e^x \\ &= e^{\ln(x)e^x}(\tfrac{1}{x}e^x + \ln(x)e^x). \end{aligned}$$

□

Exercise 0.4. I start with 5 ties to wear on special occasions. My hair-dresser tells me that my separation anxiety is because I donÕt have enough nice clothes. Taking the advice to heart, I start shopping and my tie collection starts to grow exponentially. After 3 days I have 12 ties. How long will it be before I have 100 ties? You do not need to simplify of evaluate your answer.

Solution. Our exponential model will have the form $y = ce^{kt}$. As usual c is the initial value, which in this case is 5 (ties). Thus $y = 5e^{kt}$. We also know that $y = 12$ when $t = 3$, so

$$\begin{aligned} 12 &= 5e^{3k} \\ 12/5 &= e^{3k} \\ \ln(12/5) &= 3k \\ k &= \frac{\ln(12/5)}{3}. \end{aligned}$$

Thus the number of ties is $y = 5e^{\frac{\ln(12/5)}{3}t}$. The time until 100 ties is given by solving

$$\begin{aligned}
100 &= 5e^{\frac{\ln(12/5)}{3}t} \\
20 &= e^{\frac{\ln(12/5)}{3}t} \\
\ln 20 &= \frac{\ln(12/5)}{3}t \\
t &= \frac{3\ln 20}{\ln(12/5)}.
\end{aligned}$$

□

Exercise 0.5. Which point on the graph of $y = \sqrt{x}$ is closest to the point $(4, 0)$?

Solution. The distance from (x, y) to $(4, 0)$ is

$$D = \sqrt{(x-4)^2 + y^2}.$$

Our point is constrained to lie on $y = \sqrt{x}$, so we can substitute this into the above equation to find

$$D = \sqrt{(x-4)^2 + x}.$$

To minimize this we take its derivative and find the critical points.

$$\frac{dD}{dt} = \frac{1}{2\sqrt{(x-4)^2 + x}}(2(x-4)+1)$$

This is never undefined for $x \geq 0$ (which is the only place it makes sense to ask for the graph of $y = \sqrt{x}$) and is zero precisely when

$$\begin{aligned}
2(x-4)+1 &= 0 \\
2x - 7 &= 0 \\
x &= \frac{7}{2}.
\end{aligned}$$

Thus our only critical point is $x = \frac{7}{2}$. To see that it is a minimum we can use the first derivative test: the denominator $2\sqrt{(x-4)^2 + x}$ of $\frac{dD}{dt}$ is always positive, and the numerator $2(x-4)+1$ is negative for $x < \frac{7}{2}$ and positive for $x > \frac{7}{2}$. This implies that $x = \frac{7}{2}$ is indeed a minimum, so we conclude that the closest point to $(4, 0)$ on the graph of $y = \sqrt{x}$ is the point $(\frac{7}{2}, \sqrt{\frac{7}{2}})$. □

Exercise 0.6. A salt crystal is growing is a super-saturated solution of salt. It is a perfect cube and its length, width and height are all growing at a rate of 1mm per day. What is the rate of increase of the volume of the cube when its length, width and height all equal 10 mm?

Solution. The length, width, height, and volume of a cube are related by the equation

$$V = lwh.$$

Taking an implicit derivative, and using the product rule (twice) on the right hand side, we find

$$\frac{dV}{dt} = \frac{dl}{dt}wh + l\frac{dw}{dt}h + lw\frac{dh}{dt}.$$

We're told that $l = w = h = 10$, so $V = 1000$, and $\frac{dl}{dt} = \frac{dw}{dt} = \frac{dh}{dt} = 1$; plugging all these into the above equation we have

$$\frac{dV}{dt} = (1)(10)(10) + (10)(1)(10) + (10)(10)(1) = 300.$$

□

Exercise 0.7. Evaluate $\lim_{x\to 0} \frac{\sin^2 x}{4x^2}$.

Solution. This limit has the indeterminate form $\frac{0}{0}$, so we can use l'Hôpital's rule to see

$$\lim_{x\to 0} \frac{\sin^2 x}{4x^2} = \lim_{x\to 0} \frac{2\sin x\cos x}{8x}.$$

This new limit still has the indeterminate form $\frac{0}{0}$, so using l'Hôpital's rule once more,

$$\lim_{x\to 0} \frac{2\sin x\cos x}{8x} = \lim_{x\to 0} \frac{2\cos^2 x - 2\sin^2 x}{8}.$$

Now plugging in $x = 0$ we find the limit is $\frac{1}{4}$. □

Exercise 0.8. What is $9^{\log_{49}(7)}$?

Solution. Since $7^2 = 49$, we have $7 = 49^{1/2}$, and $\log_{49}(7) = 1/2$. Thus

$$9^{\log_{49}(7)} = 9^{1/2} = \sqrt{9} = 3.$$

□

Exercise 0.9. What is $\log_4(8)$?

Solution. We have $8 = 2^3 = (2^2)^{3/2} = 4^{3/2}$, so $\log_4(8) = 3/2$. □