Midterm 1 Review Solutions

MATH 1A Fall 2015

Easier Problems

Exercise 1.1. Write down the truth tables for the following logical statements.

1. P or Q
2. P implies Q
3. P and not Q
4. (not Q) implies (not P)
5. not (P implies Q)

(Observe that some of these statements have the same truth table, and conclude that those statements are logically the same.)

Solution. P or Q:

		P: T	P: F
Q	T	T	T
	F	T	F

P implies Q:

		P: T	P: F
Q	T	T	T
	F	F	T

P and not Q:

		P: T	P: F
Q	T	F	F
	F	T	F

(not Q) implies (not P):

		P T	P F
Q	T	T	T
Q	F	F	T

not (P implies Q):

		P T	P F
Q	T	F	F
Q	F	T	F

Observe that we get the same truth tables for "P implies Q" and for "(not Q) implies (not P)", so these two statements are logically the same. This is the *contrapositive*.

We also get the same truth tables for "P and not Q" and for "not (P implies Q)", so these two are also the same, i.e. the opposite of "P implies Q" is "P and not Q". ☐

Exercise 1.2. Prove that for every $a \in \mathbb{R}$, we have $|a| \geq a$.

Proof. Recall that the absolute value $|a|$ is defined to be a or $-a$, whichever is positive. If a is positive, then $|a| \geq a$ because in fact $|a| = a$. (Same if $a = 0$). If a is negative, $|a| \geq a$ because $|a|$ is positive and a is negative. ☐

Exercise 1.3. The sum law for limits:

Suppose f, g are real-valued functions, and suppose that

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} g(x) = M.$$

Then

$$\lim_{x \to a} f(x) + g(x) = L + M.$$

Proof. We want to get

$$|(f(x) + g(x)) - (L + M)| < \varepsilon,$$

and by the triangle inequality

$$|(f(x) + g(x)) - (L + M)| < |f(x) - L| + |g(x) - M|$$

so it's enough to make

$$|f(x) - L| + |g(x) - M| < \varepsilon.$$

We'll do this by making $|f(x) - L| < \varepsilon/2$ and $|g(x) - M| < \varepsilon/2$.

So, here's the proof:

Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there is a $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f(x) - L| < \varepsilon/2$. Similarly, since $\lim_{x\to a} g(x) = M$, there is a $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $|g(x) - M| < \varepsilon/2$.

Set $\delta = \min(\delta_1, \delta_2)$, and suppose $0 < |x-a| < \delta$. Then $0 < |x-a| < \delta_1$ so $|f(x) - L| < \varepsilon/2$, and $0 < |x-a| < \delta_2$ so $|g(x) - M| < \varepsilon/2$. Now

$$|(f(x) + g(x)) - (L+M)| = |(f(x) - L) + (g(x) - M)|$$
$$\leq |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We've shown $|(f(x) + g(x)) - (L+M)| < \varepsilon$, so we conclude $\lim_{x\to a} f(x) + g(x) = L + M$. □

Exercise 1.4. Suppose $|x-3| \leq 2$. Conclude that $|x+1| \leq 6$.

Solution.

$$\begin{aligned} |x-3| &\leq 2 \\ -2 \leq x-3 &\leq 2 \\ 2 \leq x+1 &\leq 6 \\ |x+1| &\leq 6 \end{aligned}$$

□

Exercise 1.5. The constant multiple law for limits:

Suppose f is a real-valued function and $c \in \mathbb{R}$. Suppose also that

$$\lim_{x\to a} f(x) = L.$$

Then

$$\lim_{x\to a} cf(x) = cL.$$

Solution. Note that if $c = 0$ it's trivial, so we can suppose $c \neq 0$.

Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon/|c|$, so

$$|cf(x) - cL| = |c||f(x) - L| < \varepsilon,$$

as desired. □

Exercise 1.6. Suppose $|x-1| \leq 4$. Find a bound for $|x-7|$.

Solution.

$$\begin{aligned} |x-1| &\leq 4 \\ -4 \leq x-1 &\leq 4 \\ -10 \leq x-7 &\leq -2 \\ |x-7| &\leq 10 \end{aligned}$$

□

Exercise 1.7. Define what it means to say $\lim_{x\to a^+} f(x) = \infty$. Then show $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$. What is $\lim_{x\to 1} \frac{1}{x-1}$?

Solution. We say $\lim_{x\to a^+} f(x) = \infty$ if for all $M \in \mathbb{R}$ there is a $\delta > 0$ such that if $a < x < a + \delta$ then $f(x) > M$.

Let $M \in \mathbb{R}$. Without loss of generality we can assume $M > 0$. Set $\delta = \frac{1}{M}$. Suppose $1 < x < 1 + \delta$, i.e. $1 < x < 1 + \frac{1}{M}$. Then $0 < x - 1 < \frac{1}{M}$, so $\frac{1}{x-1} > M$. Thus $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$.

A similar argument shows that $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$, so $\lim_{x\to 1} \frac{1}{x-1}$ does not exist (even as an infinite limit). $\square$

Exercise 1.8. The difference law for limits:

Suppose f, g are real-valued functions, and suppose that

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a} f(x) - g(x) = L - M.$$

Proof. Just the same as the sum law, with minus signs interted carefully.

Let $\varepsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there is a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $|f(x) - L| < \varepsilon/2$. Similarly, since $\lim_{x\to a} g(x) = M$, there is a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|g(x) - M| < \varepsilon/2$.

Set $\delta = \min(\delta_1, \delta_2)$, and suppose $0 < |x - a| < \delta$. Then $0 < |x - a| < \delta_1$ so $|f(x) - L| < \varepsilon/2$, and $0 < |x - a| < \delta_2$ so $|g(x) - M| < \varepsilon/2$. Now

$$\begin{aligned} |(f(x) - g(x)) - (L - M)| &= |(f(x) - L) - (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We've shown $|(f(x) - g(x)) - (L - M)| < \varepsilon$, so we conclude $\lim_{x\to a} f(x) - g(x) = L - M$. $\square$

Exercise 1.9. Define what it means to say $\lim_{x\to a^-} f(x) = L$. Then show $\lim_{x\to 2^-} \frac{x-2}{|x-2|} = -1$. What is $\lim_{x\to 2^+} \frac{x-2}{|x-2|}$?

Solution. We say $\lim_{x\to a^-} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $a - \delta < x < a$ then $|f(x) - L| < \varepsilon$.

Let $\varepsilon > 0$. Choose any $\delta > 0$, it doesn't matter what. Suppose $2 - \delta < x < 2$. Since $x < 2$ we have $\frac{x-2}{|x-2|} = -1$, so

$$\left| \frac{x-2}{|x-2|} - (-1) \right| = 0 < \varepsilon.$$

$\square$

Exercise 1.10. Define what it means to say $\lim_{x\to\infty} f(x) = L$. Then show $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

Solution. We say $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that if $x > N$ then $|f(x) - L| < \varepsilon$.

Let $\varepsilon > 0$, and set $N = \frac{1}{\sqrt{\varepsilon}}$. Suppose $x > N$, i.e. $x > \frac{1}{\sqrt{\varepsilon}}$. Then $0 < \frac{1}{x} < \sqrt{\varepsilon}$, so $0 < \frac{1}{x^2} < \varepsilon$, and $|\frac{1}{x^2}| < \varepsilon$. $\square$

Exercise 1.11. State the definition of continuity. Then prove that $f(x) = 10x$ is continuous.

Solution. A real-valued function is continuous if $\lim_{x\to a} f(x) = f(a)$ for all a in the domain.

Thus to prove $f(x) = 10x$ is continuous, we want to show $\lim_{x\to a} 10x = 10a$ for all a. Let $\varepsilon > 0$, and set $\delta = \varepsilon/10$. Suppose $0 < |x-a| < \delta = \varepsilon/10$. Then

$$|10x - 10a| = 10|x-a| < 10 \cdot \varepsilon/10 = \varepsilon.$$

□

Exercise 1.12. Decide whether the following statements are true or false.

1. If f is continuous at a, then f is differentiable at a.
2. If f is differentiable at a, then f is continuous at a.
3. If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist, then $\lim_{x\to a} f(x)$ exists.

Solution.

1. False, e.g. $|x|$ is continuous but not differentiable at $x = 0$.
2. True (see the Harder problems for a proof).
3. False, the one-sided limits must exist and also agree in order for the two-sided limit to exist.

□

Exercise 1.13. Prove that

$$\lim_{x\to 0} \frac{x}{\cos x} = 0.$$

Proof. Note that $\lim_{x\to 0} \cos x = 1$ and $\lim x \to 0 x = 0$, so by the quotient law $\lim_{x\to 0} \frac{x}{\cos x} = 0$.

Alternatively, if we don't want to assume that cos is continuous (as we had to do to evaluate the first limit), we can bound $\frac{1}{\cos x}$, for example $1 \le \frac{1}{\cos x} \le \sqrt{2}$ on the interval $(-\pi/4, \pi/4)$, and then use the squeeze theorem. □

Exercise 1.14. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x\to 3} x^2 - 2x + 1 = 4.$$

Proof. Let $\varepsilon > 0$ and choose $\delta = \min(1, \varepsilon/5)$. Suppose $0 < |x-3| < \delta$. Then

$$\begin{gathered} |x-3| < 1 \\ -1 < x-3 < 1 \\ 3 < x+1 < 5 \\ |x+1| < 5; \end{gathered}$$

and $|x-3| < \varepsilon/5$. Now

$$\begin{gathered} |x^2 - 2x + 1 - 4| \\ = |x^2 - 2x - 3| \\ = |(x+1)(x-3)| \\ = |x+1||x-3| \\ < 5 \cdot \varepsilon/5 = \varepsilon. \end{gathered}$$

□

Exercise 1.15. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x\to 1} 2x^2 - 3 = -1.$$

Proof. Let $\varepsilon > 0$ and choose $\delta = \min(1, \varepsilon/6)$. Suppose $0 < |x-1| < \delta$. Then

$$\begin{aligned}
|x-1| &< 1 \\
-1 < x - 1 &< 1 \\
1 < x + 1 &< 3 \\
|x+1| &< 3;
\end{aligned}$$

and $|x-1| < \varepsilon/6$. Now

$$\begin{aligned}
&|2x^2 - 3 - (-1)| \\
&= |2x^2 - 2| \\
&= 2|x^2 - 1| \\
&= 2|(x+1)(x-1)| \\
&= 2|x+1||x-1| \\
&< 2 \cdot 3 \cdot \varepsilon/6 = \varepsilon.
\end{aligned}$$

□

Exercise 1.16. The squeeze theorem:

Let f, g, h be real-valued functions and $a \in \mathbb{R}$. Suppose that when x is near a, except possibly at a, we have $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x).$$

Then

$$\lim_{x\to a} g(x) = L.$$

Proof. Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $f(x) \leq g(x) \leq h(x)$.

Since $\lim_{x\to a} f(x) = L$, we can choose $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $|f(x) - L| < \varepsilon$. Similarly, since $\lim_{x\to a} h(x) = L$, we can choose $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3$ then $|h(x) - L| < \varepsilon$.

Set $\delta = \min(\delta_1, \delta_2, \delta_3)$, and suppose $0 < |x-a| < \delta$.

Then $0 < |x-a| < \delta_1$, so $f(x) \leq g(x) \leq h(x)$.

Also $0 < |x-a| < \delta_2$, so $|f(x) - L| < \varepsilon$, i.e. $L - \varepsilon < f(x) < L + \varepsilon$.

Also $0 < |x-a| < \delta_3$, so $|h(x) - L| < \varepsilon$, i.e. $L - \varepsilon < h(x) < L + \varepsilon$.

Combining these, we see

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon,$$

so $|g(x) - L| < \varepsilon$. □

Harder Problems

Exercise 2.1. Prove the following sort-of-generalization of the sqeeze theorem.

Let f, g, h be real-valued functions, and $a \in \mathbb{R}$. Suppose when x is near a, except possibly at a, these functions satisfy $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M, \qquad \lim_{x\to a} h(x) = N.$$

Then

$$L \leq M \leq N.$$

Solution. Observe that it's enough to prove if $f(x) \leq g(x)$ then $L \leq M$, because the same proof applied to $g(x) \leq h(x)$ will show $M \leq N$.

Recall also from the first Easier Problem that "if P then Q", in our case "if $f(x) \leq g(x)$ near a then $L \leq M$", is logically equivalent to "if (not Q) then (not P)", in our case "if $L > M$ then $f(x) \not\leq g(x)$ near a". Since it's all the same, we'll prove the latter instead. In fact, we'll prove the stronger statement that "if $L > M$ then $f(x) > g(x)$ near a".

Let $\varepsilon = L - M > 0$. Note that $L - \varepsilon/2 = M + \varepsilon/2$.

Since $\lim_{x\to a} f(x) = L$, there is a $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f(x) - L| < \varepsilon/2$. Similarly, since $\lim_{x\to a} g(x) = M$, there is a $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $|g(x) - M| < \varepsilon/2$.

Set $\delta = \min(\delta_1, \delta_2)$, and suppose $0 < |x-a| < \delta$. Then $0 < |x-a| < \delta_1$, so

$$|f(x) - L| < \varepsilon/2$$
$$L - \varepsilon/2 < f(x) < L + \varepsilon/2.$$

Also $0 < |x-a| < \delta_2$, so

$$|g(x) - M| < \varepsilon/2$$
$$M - \varepsilon/2 < g(x) < M + \varepsilon/2.$$

Combining these, we see

$$g(x) < M + \varepsilon/2 = L - \varepsilon/2 < f(x),$$

so $g(x) < f(x)$ for all x with $0 < |x-a| < \delta$. Thus we've shown that if $L > M$, then $f(x) > g(x)$ near a. □

Exercise 2.2. Show that there is always a pair of diametrically opposite points on Earth's equator where the temperature at both points is the same.

Proof. Let $T(x)$ be the function that gives the temperature at a point x on the equator, and denote by $-x$ the diametrically opposite point. Consider the function $f(x) = T(x) - T(-x)$, i.e. the difference in temperature between a point and its opposite. Note that temperature is a continuous function, and so $f(x)$ is continuous as well. Note also that a point x has our desired property, i.e. the same temperature as its opposite point, precisely when $f(x) = 0$.

If there is no point where the temeperature differs, i.e. if $f(x) = 0$ for all x, then of course we're done; any point has the desired property.

On the other hand, suppose there is a point where $f(x)$ is non-zero, say $f(x_0) = t$. Then $f(-x_0) = T(-x_0) - T(x_0) = -f(x_0) = -t$. One of $t, -t$ is strictly positive and the other strictly negative, so by the intermediate value theorem we conclude there is a point between x_0 and $-x_0$ where $f(x) = 0$, and this point has our desired property. □

Exercise 2.3. Prove the following variation of the squeeze theorem:

Let f, g, h be real-valued functions. Suppose there exists $N > 0$ such that for all $x > N$, we have $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x\to\infty} f(x) = L = \lim_{x\to\infty} h(x).$$

Then

$$\lim_{x\to\infty} g(x) = L.$$

Proof. Let $\varepsilon > 0$.

Choose $N_1 \in \mathbb{R}$ such that if $x > N$ then $f(x) \leq g(x) \leq h(x)$.

Since $\lim_{x\to\infty} f(x) = L$, we can choose $N_2 \in \mathbb{R}$ such that if $x > N_2$ then $|f(x) - L| < \varepsilon$. Similarly, since $\lim_{x\to\infty} h(x) = L$, we can choose $N_3 > 0$ such that if $x > N_3$ then $|h(x) - L| < \varepsilon$.

Set $N = \max(N_1, N_2, N_3)$, and suppose $x > N$.

Then $x > N_1$, so $f(x) \leq g(x) \leq h(x)$.

Also $x > N_2$, so $|f(x) - L| < \varepsilon$, i.e. $L - \varepsilon < f(x) < L + \varepsilon$.

Also $x > N_3$, so $|h(x) - L| < \varepsilon$, i.e. $L - \varepsilon < h(x) < L + \varepsilon$.

Combining these, we see

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon,$$

so $|g(x) - L| < \varepsilon$. □

Exercise 2.4. The squeeze theorem for limits of the form $\lim_{x\to a^+} f(x) = L$:

Let f, g, h be real-valued functions and $a \in \mathbb{R}$. Suppose that when x is near to and greater than a, we have $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x\to a^+} f(x) = L = \lim_{x\to a^+} h(x).$$

Then

$$\lim_{x\to a^+} g(x) = L.$$

Proof. Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ such that if $a < x < a + \delta_1$ then $f(x) \leq g(x) \leq h(x)$.

Since $\lim_{x\to a^+} f(x) = L$, we can choose $\delta_2 > 0$ such that if $a < x < a + \delta_2$ then $|f(x) - L| < \varepsilon$. Similarly, since $\lim_{x\to a^+} h(x) = L$, we can choose $\delta_3 > 0$ such that if $a < x < a + \delta_3$ then $|h(x) - L| < \varepsilon$.

Set $\delta = \min(\delta_1, \delta_2, \delta_3)$, and suppose $a < x < a + \delta$.

Then $a < x < a + \delta_1$, so $f(x) \leq g(x) \leq h(x)$.

Also $a < x < a + \delta_2$, so $|f(x) - L| < \varepsilon$, i.e. $L - \varepsilon < f(x) < L + \varepsilon$.

Also $a < x < a + \delta_3$, so $|h(x) - L| < \varepsilon$, i.e. $L - \varepsilon < h(x) < L + \varepsilon$.

Combining these, we see

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon,$$

so $|g(x) - L| < \varepsilon$. □

Exercise 2.5. Show that the sum of two continuous functions is continuous.

Proof. Suppose f, g are continuous functions, i.e.

$$\lim_{x\to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x\to a} g(x) = g(a)$$

for all a. By the sum law for limits,

$$\lim_{x\to a} f(x) + g(x) = f(a) + g(a)$$

for all a, i.e. $f(x) + g(x)$ is continuous. □

Exercise 2.6. Prove the following variation of the sum law for limits:

Let f, g be real valued functions, and $a, L, M \in \mathbb{R}$. Suppose that

$$\lim_{x\to a^+} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^+} g(x) = M.$$

Then

$$\lim_{x\to a^+} f(x) + g(x) = L + M.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to a^+} f(x) = L$, there is a $\delta_1 > 0$ such that if $a < x < a + \delta_1$ then $|f(x) - L| < \varepsilon/2$. Similarly, since $\lim_{x\to a^+} g(x) = M$, there is a $\delta_2 > 0$ such that if $a < x < a + \delta_2$ then $|g(x) - M| < \varepsilon/2$.

Set $\delta = \min(\delta_1, \delta_2)$, and suppose $a < x < a + \delta$. Then $a < x < a + \delta_1$ so $|f(x) - L| < \varepsilon/2$, and $a < x < a + \delta_2$ so $|g(x) - M| < \varepsilon/2$. Now

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We've shown $|(f(x) + g(x)) - (L + M)| < \varepsilon$, so we conclude $\lim_{x\to a^+} f(x) + g(x) = L + M$. □

Exercise 2.7. The difference law for limits of the form $\lim_{x\to a^-} f(x) = L$:

Suppose f, g are real-valued functions, and suppose that

$$\lim_{x\to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^-} g(x) = M.$$

Then

$$\lim_{x\to a^-} f(x) - g(x) = L - M.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to a^-} f(x) = L$, there is a $\delta_1 > 0$ such that if $a - \delta_1 < x < a$ then $|f(x) - L| < \varepsilon/2$. Similarly, since $\lim_{x\to a^-} g(x) = M$, there is a $\delta_2 > 0$ such that if $a - \delta_2 < x < a$ then $|g(x) - M| < \varepsilon/2$.

Set $\delta = \min(\delta_1, \delta_2)$, and suppose $a - \delta < x < a$. Then $a - \delta_1 < x < a$ so $|f(x) - L| < \varepsilon/2$, and $a - \delta_2 < x < a$ so $|g(x) - M| < \varepsilon/2$. Now

$$\begin{aligned} |(f(x) - g(x)) - (L - M)| &= |(f(x) - L) - (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We've shown $|(f(x) - g(x)) - (L - M)| < \varepsilon$, so we conclude $\lim_{x\to a^-} f(x) - g(x) = L - M$. □

Exercise 2.8. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x\to 2} x^3 - x^2 + 2x + 1 = 9.$$

Proof. Let $\varepsilon > 0$. Set $\delta = \min(1, \varepsilon/16)$.

Suppose $|x-2| < \delta$. Then $|x-2| < 1$, so

$$-1 < x - 2 < 1$$
$$1 < x < 3$$

and since $x^2 + x + 4$ is increasing on the interval $[1,3]$, we have

$$1^2 + 1 + 4 = 6 < x^2 + x + 4 < 16 = 3^2 + 3 + 4$$
$$|x^2 + x + 4| < 16.$$

Also $|x-2| < \varepsilon/16$.

Now

$$\begin{aligned}
&|x^3 - x^2 + 2x + 1 - 9| \\
=&|x^3 - x^2 + 2x - 8| \\
=&|(x-2)(x^2 + x + 4)| \\
=&|x-2||x^2 + x + 4| \\
<&\frac{\varepsilon}{16} \cdot 16 \\
=&\varepsilon.
\end{aligned}$$

Thus $|x^3 - x^2 + 2x + 1 - 9| < \varepsilon$, as desired. □

Exercise 2.9. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x \to 2} x^4 - 12 = 4.$$

Proof. Let $\varepsilon > 0$, and set $\delta = \min(1, \varepsilon/203)$.

Suppose $|x-2| < \delta$. Then $|x-4| < 1$, so

$$-1 < x - 4 < 1$$
$$5 < x + 2 < 7$$
$$|x + 2| < 7$$

and

$$-1 < x - 4 < 1$$
$$3 < x < 5$$
$$0 < x^2 < 25$$
$$4 < x^2 + 4 < 29$$
$$|x^2 + 4| < 29.$$

Also $|x-4| < \varepsilon/203$.

Now

$$
\begin{aligned}
&|x^4 - 12 - 4| \\
=&|x^4 - 16| \\
=&|(x^2-4)(x^2+4)| \\
=&|(x-2)(x+2)(x^2+4)| \\
=&|(x-2)||(x+2)||(x^2+4)| \\
<&\frac{\varepsilon}{203} \cdot 7 \cdot 29 \\
=&\varepsilon.
\end{aligned}
$$

Thus $|x^4 - 12 - 4| < \varepsilon$, as desired. □

Exercise 2.10. Suppose $|x-a| < \delta$. Find a bound for $|x-b|$ (which may depend on a, b, δ).

Solution.

$$
\begin{gathered}
|x-a| < \delta \\
-\delta < x-a < \delta \\
-\delta + a - b < x - b < \delta + a - b \\
|x-b| < \max(|-\delta + a - b|, |\delta + a - b|)
\end{gathered}
$$

□

Exercise 2.11. Prove that if a function f is differentiable at a (i.e. if the limit defining the derivative at a exists) then f is continuous at a.

Proof. Suppose f is differentiable at a. Then the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. Now observe, using the product law for limits,

$$
\begin{aligned}
\lim_{x\to a} f(x) - f(a) = \lim_{h\to 0} f(a+h) - f(a) &= \lim_{h\to 0} h \cdot \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} h \cdot \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = 0 \cdot f'(a) = 0.
\end{aligned}
$$

Since $\lim_{x\to a} f(x) - f(a) = 0$, we conclude that $\lim_{x\to a} f(x) = f(a)$, i.e. f is continuous at a. □

Exercise 2.12. Find the derivative of x^x. [Hint: be careful trying to apply the chain rule here: write down precisely what f and g are, and you'll probably find that it doesn't work! The key is to rewrite x^x in a form that's easier to handle. Use the fact that $x = e^{\log x}$.]

Solution. Since $x = e^{\log x}$, we can rewrite $x^x = (e^{\log x})^x = e^{x \log x}$, and this looks more like something we can evaluate. We have

$$\begin{aligned}
&\frac{d}{dx} x^x \\
=&\frac{d}{dx} e^{x \log x} \\
=&e^{x \log x} \frac{d}{dx} x \log x \\
=&e^{x \log x} (\log x + x \cdot \tfrac{1}{x}) \\
=&x^x (\log x + 1).
\end{aligned}$$

□

Bonus Material: Stewart Chapter 2 Review Exercises #6-9, 15-20

Here are a bunch of problems on evaluating limits from the Chapter 2 Review of the textbook. I won't prove anything, just give a quick explanation of what the limit is and why.

Exercise 2.6.

$$\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3}$$

Solution. Factor and cancel:

$$\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3} = \lim_{x\to 1^+} \frac{(x+3)(x-3)}{(x-1)(x+3)} = \lim_{x\to 1^+} \frac{x-3}{x-1} = -\infty.$$

As x approaches 1 from above, $\frac{1}{x-1}$ will go to $+\infty$, and $x-3$ will be about -2, so the limit is $-\infty$. □

Exercise 2.7.

$$\lim_{h\to 0} \frac{(h-1)^3+1}{h}.$$

Solution. Expand the cube:

$$\lim_{h\to 0} \frac{(h-1)^3+1}{h} = \lim_{h\to 0} \frac{h^3-3h^2+3h-1+1}{h} = \lim_{h\to 0} \frac{h^3-3h^2+3h}{h} = \lim_{h\to 0} h^2-3h+3 = 3.$$

□

Exercise 2.8.

$$\lim_{t\to 2} \frac{t^2-4}{t^3-8}$$

Solution. Factor and cancel:

$$\lim_{t\to 2} \frac{t^2-4}{t^3-8} = \lim_{t\to 2} \frac{(t-2)(t+2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2} \frac{(t+2)}{(t^2+2t+4)} = \frac{1}{3}.$$

□

Exercise 2.9.

$$\lim_{r\to 9} \frac{\sqrt{r}}{(r-9)^4}$$

Solution. Plug in: as r approaches 9 (from either side), $\frac{1}{(r-9)^4}$ goes to $+\infty$, and $\sqrt{r}$ is about 3, so the limit is $+\infty$. □

Exercise 2.15.

$$\lim_{x\to \pi^-} \ln(\sin x)$$

Solution. Composition: set $y=\sin x$, so $\lim_{x\to\pi^-}\ln(\sin x) = \lim_{x\to\pi^-}\ln y$. As x approaches π from below, $y=\sin x$ approaches 0 from above. As y approaches 0 from above, $\ln y$ approaches $-\infty$. Thus the limit is $-\infty$. □

Exercise 2.16.

$$\lim_{x\to -\infty} \frac{1-2x^2-x^4}{5+x-3x^4}$$

Solution. Highest powers: since we're taking a limit at $-\infty$, we can ignore everything except the highest power in the numerator and denominator. That is,

$$\lim_{x\to-\infty} \frac{1-2x^2-x^4}{5+x-3x^4} = \lim_{x\to-\infty} \frac{-x^4}{-3x^4} = \frac{1}{3}.$$

□

Exercise 2.17.

$$\lim_{x\to\infty} (\sqrt{x^2+4x+1} - x)$$

Solution. Not sure about this one. Let me know if y'all have ideas. WolframAlpha tells me the limit is 2. □

Exercise 2.18.

$$\lim_{x\to\infty} e^{x-x^2}$$

Solution. Composition: let $y = x - x^2$, so $\lim_{x\to\infty} e^{x-x^2} = \lim_{x\to\infty} e^y$. As x approaches ∞, $y = x - x^2$ approaches $-\infty$. As y approaches $-\infty$, e^y approaches 0. Thus the limit is 0. □

Exercise 2.19.

$$\lim_{x\to 0^+} \arctan(1/x)$$

Solution. Composition: let $y = 1/x$, so $\lim_{x\to 0^+} \arctan(1/x) = \lim_{x\to 0^+} \arctan(y)$. As x approaches 0 from above, $y = 1/x$ approaches $+\infty$. As y approaches $+\infty$, $\arctan(y)$ approaches $\pi/2$. Thus the limit is $\pi/2$. □

Exercise 2.20.

$$\lim_{x\to 1} \left(\frac{1}{x-1} + \frac{1}{x^2-3x+2}\right)$$

Solution. Common denominator and cancel:

$$\lim_{x\to 1} \left(\frac{1}{x-1} + \frac{1}{x^2-3x+2}\right) = \lim_{x\to 1} \left(\frac{1}{x-1} + \frac{1}{(x-1)(x-2)}\right)$$

$$= \lim_{x\to 1} \left(\frac{x-2}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)}\right) = \lim_{x\to 1} \left(\frac{x-2+1}{(x-1)(x-2)}\right)$$

$$= \lim_{x\to 1} \left(\frac{x-1}{(x-1)(x-2)}\right) = \lim_{x\to 1} \left(\frac{1}{x-2}\right) = -1.$$

□