Midterm 1 Review

MATH 1A Fall 2015

Easier Problems

Exercise 1.1. Write down the truth tables for the following logical statements.

1. P or Q
2. P implies Q
3. P and not Q
4. not Q implies not P
5. not (P implies Q)

(Observe that some of these statements have the same truth table, and conclude that those statements are logically the same.)

Exercise 1.2. Prove that for every $a \in \mathbb{R}$, we have $|a| \geq a$.

Exercise 1.3. State and prove the sum law for limits.

Exercise 1.4. Suppose $|x-3| \leq 2$. Conclude that $|x+1| \leq 6$.

Exercise 1.5. State and prove the constant multiple law for limits.

Exercise 1.6. Suppose $|x-1| \leq 4$. Find a bound for $|x-7|$.

Exercise 1.7. Define what it means to say $\lim_{x\to a^+} f(x) = \infty$. Then show $\lim_{x\to 1^+} \frac{1}{x-1} = \infty$. What is $\lim_{x\to 1} \frac{1}{x-1}$?

Exercise 1.8. State and prove the difference law for limits.

Exercise 1.9. Define what it means to say $\lim_{x\to a^-} f(x) = L$. Then show $\lim_{x\to 2^-} \frac{x-2}{|x-2|} = -1$. What is $\lim_{x\to 2^+} \frac{x-2}{|x-2|}$?

Exercise 1.10. Define what it means to say $\lim_{x\to\infty} f(x) = L$. Then show $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

Exercise 1.11. State the definition of continuity. Then prove that $f(x) = 10x$ is continuous.

Exercise 1.12. Decide whether the following statements are true or false.

1. If f is continuous at a, then f is differentiable at a.
2. If f is differentiable at a, then f is continuous at a.
3. If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist, then $\lim_{x\to a} f(x)$ exists.

Exercise 1.13. Prove that

$$\lim_{x\to 0} \frac{x}{\cos x} = 0.$$

Exercise 1.14. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x \to 3} x^2 - 2x + 1 = 4.$$

Exercise 1.15. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x \to 1} 2x^2 - 3 = -1.$$

Exercise 1.16. State and prove the squeeze theorem.

Harder Problems

Exercise 2.1. Prove the following sort-of-generalization of the sqeeze theorem.

Let f, g, h be real-valued functions, and $a \in \mathbb{R}$. Suppose when x is near a, except possibly at a, these functions satisfy $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x \to a} f(x) = L, \qquad \lim_{x \to a} g(x) = M, \qquad \lim_{x \to a} h(x) = N.$$

Then

$$L \leq M \leq N.$$

Exercise 2.2. Show that there is always a pair of diametrically opposite points on Earth's equator where the temperature at both points is the same.

Exercise 2.3. Prove the following variation of the squeeze theorem:

Let f, g, h be real-valued functions. Suppose there exists $N > 0$ such that for all $x > N$, we have $f(x) \leq g(x) \leq h(x)$. Suppose also that

$$\lim_{x \to \infty} f(x) = L = \lim_{x \to \infty} h(x).$$

Then

$$\lim_{x \to \infty} g(x) = L.$$

Exercise 2.4. State and prove the squeeze theorem for limits of the form $\lim_{x \to a^+} f(x) = L$.

Exercise 2.5. Show that the sum of two continuous functions is continuous.

Exercise 2.6. Prove the following variation of the sum law for limits:

Let f, g be real valued functions, and $a, L, M \in \mathbb{R}$. Suppose that

$$\lim_{x \to a^+} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} g(x) = M.$$

Then

$$\lim_{x \to a^+} f(x) + g(x) = L + M.$$

Exercise 2.7. State and prove the difference law for limits of the form $\lim_{x \to a^-} f(x) = L$.

Exercise 2.8. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x \to 2} x^3 - x^2 + 2x + 1 = 9.$$

Exercise 2.9. Prove using the definition of a limit (i.e. ε and δ) that

$$\lim_{x \to 2} x^4 - 12 = 4.$$

Exercise 2.10. Suppose $|x - a| < \delta$. Find a bound for $|x - b|$ (which may depend on a, b, δ).

Exercise 2.11. Prove that if a function f is differentiable at a (i.e. if the limit defining the derivative at a exists) then f is continuous at a.

Exercise 2.12. Find the derivative of x^x. [Hint: be careful trying to apply the chain rule here: write down precisely what f and g are, and you'll probably find that it doesn't work! The key is to rewrite x^x in a form that's easier to handle. Use the fact that $x = e^{\log x}$.]

Bonus Material: Stewart Chapter 2 Review Exercises #6-9, 15-20

Here are a bunch of problems on evaluating limits from the Chapter 2 Review of the textbook. No need to prove anything, just give a quick explanation of what the limit is and why.

Exercise 2.6.

$$\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3}$$

Exercise 2.7.

$$\lim_{h\to 0} \frac{(h-1)^3+1}{h}.$$

Exercise 2.8.

$$\lim_{t\to 2} \frac{t^2-4}{t^3-8}$$

Exercise 2.9.

$$\lim_{r\to 9} \frac{\sqrt{r}}{(r-9)^4}$$

Exercise 2.15.

$$\lim_{x\to \pi^-} \ln(\sin x)$$

Exercise 2.16.

$$\lim_{x\to -\infty} \frac{1-2x^2-x^4}{5+x-3x^4}$$

Exercise 2.17.

$$\lim_{x\to\infty} (\sqrt{x^2+4x+1}-x)$$

Exercise 2.18.

$$\lim_{x\to\infty} e^{x-x^2}$$

Exercise 2.19.

$$\lim_{x\to 0^+} \arctan(1/x)$$

Exercise 2.20.

$$\lim_{x\to 1} \left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)$$