Indecomposable objects of $\underline{\mathrm{Rep}}(S_t)$

Recall that $\underline{\mathrm{Rep}}_0(S_t)$ was a category "freely generated" by an object V of ~~????~~ dimension t, and morphisms $V^{\otimes m} \to V^{\otimes n}$ were given by the $\mathbb{C}$-linear space with basis given by partition diagrams of size (m, n) (eg  $m=3, n=4$) though people may use other conventions for top/bottom)
In particular, $\mathrm{End}(V^{\otimes n}) = \mathrm{Par}_n(t)$, the partition algebra of size n with parameter t, which is semisimple whenever $t \notin \{0, 1, 2, \ldots, n-1\}$.
We write $[n]$ for $V^{\otimes n}$.
$\underline{\mathrm{Rep}}(S_t)$ was defined as the Karoubian envelope of $\underline{\mathrm{Rep}}_0(S_t)$ (i.e. additive envelope followed by idempotent completion). General facts about Karoubian envelopes imply that $\underline{\mathrm{Rep}}(S_t)$ is a Krull-Schmidt category, i.e. that if

$$N_1 \oplus N_2 \oplus \cdots \oplus N_\ell \cong N_1' \oplus N_2' \oplus \cdots \oplus N_{\ell'}'$$

where N_i, N_j' are indecomposable, then in fact $\ell = \ell'$ and $N_i \cong N'_{\sigma(i)}$ for some permutation σ.
Objects of $\underline{\mathrm{Rep}}(S_t)$ are of the form $([n_1] \oplus \cdots \oplus [n_i], e)$ where e is an idempotent in $\mathrm{End}([n_1] \oplus \cdots \oplus [n_i])$, and such an object is indecomposable iff e is primitive (i.e. $e = e_1 + e_2$ with $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 e_2 = e_2 e_1 = 0$ implies $e_1 = 0$ or $e_2 = 0$).
Since $([n_1] \oplus \cdots \oplus [n_i], e)$ is a summand of $[n_1] \oplus \cdots \oplus [n_i]$
$\left(([n_1] \oplus \cdots \oplus [n_i], 1) \cong ([n_1] \oplus \cdots \oplus [n_i], e) \oplus ([n_1] \oplus \cdots \oplus [n_i], 1-e)\right)$
the Krull-Schmidt property tells us, each indecomposable is a summand of $[n_1] \oplus \cdots \oplus [n_i]$, and hence of some $[n]$.

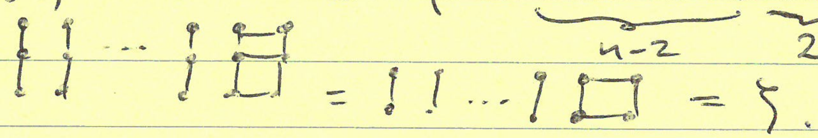
So, it's enough to understand direct summands of $[n]$, i.e. idempotents in $\mathrm{End}([n]) = \mathrm{Par}_n(t)$.

We use the following fact from noncommutative algebra. If A is a f.d. alg (over a field), ξ an idempotent then we have a bijection:

$$\left\{\begin{array}{l}\text{primitive idempotents}\\ \text{in } A \text{ up to conj.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{primitive idempotents}\\ \text{in } A/(\xi) \text{ up to conj}\end{array}\right\} \amalg \left\{\begin{array}{l}\text{primitive idempotents}\\ \text{of } \xi A \xi \text{ up to conj.}\end{array}\right\}$$

(Recall that conjugacy of idempotents means conjugacy by an element of $A^\times$, and two idempotents are conjugate iff they generate isomorphic A-submodules of A)

In our case, we take $\xi =$  $\in \mathrm{Par}_n(t)$

Note: $\xi^2 =$  $= \xi$.

We also have $\mathbb{C}S_n \hookrightarrow \mathrm{Par}_n(t)$ via the obvious embedding $\sigma \mapsto$ partition diagram where i in the top row is connected to $\sigma(i)$ in the bottom row only

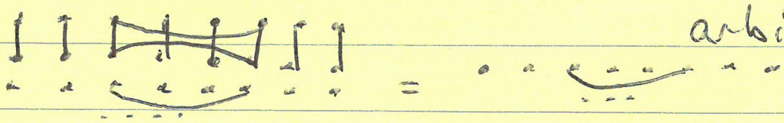
eg. $(132) \mapsto$ 

If a partition diagram π isn't a permutation, then either
(a) π has two vertices on the same side in the same part (this happens whenever there's a part with 3 or more vertices by the pigeonhole principle)
(b) there is at least one vertex in a part of its own.

We WLOG that these features occur on the top row.

In case (a):

$\pi =$  (unconnected nodes connected arbitrarily)

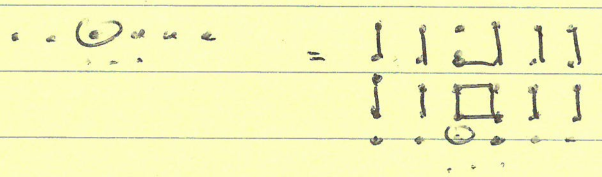
$(\sigma \rho \sigma^{-1})$ 

$\Rightarrow \pi' =$

here σ is a permutation taking $n-1, n$ to the indices of the two vertices in the same component. This shows $\pi \in (\rho)$.

In case (b):

$\pi =$  (connections arbitrary except circled vertex is unconnected)

$\Rightarrow \pi =$ 

← some partition (not important)

← conjugate to ρ

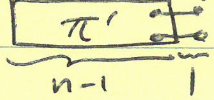
← π.

So again $\pi \in (\rho)$.

So it's easy to check that we have a S.E.S.

$$0 \to (\rho) \longrightarrow Par_n(t) \longrightarrow \mathbb{C}S_n \longrightarrow 0$$

In particular $Par_n(t)/(\rho) \cong \mathbb{C}S_n$, and the primitive idempotents (up to conj.) of $\mathbb{C}S_n$ are labelled by partitions $\lambda \vdash n$ (but we won't elaborate)

Note also that $\rho Par_n(t) \rho$ is spanned by diagrams of the form $\pi =$ , since applying ρ on either side forces the last vertex to be in the same part as the second last vertex (for both top and bottom). Again, it's easy to see that $\rho Par_n(t) \rho \cong Par_{n-1}(t)$.

Using our lemma about primitive idempotents, we have

Prop^{n} If $t \neq 0$ $\left\{ \begin{array}{l} \text{Primitive idempotents} \\ \text{in } \mathrm{Par}_n(t) \text{ up to conj.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{partitions } \lambda \\ 0 \leq |\lambda| \leq n \end{array} \right\}$

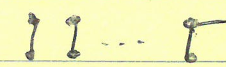
If $t = 0$ $\left\{ \begin{array}{l} \text{primitive idempotents} \\ \text{in } \mathrm{Par}_n(t) \text{ up to conj} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{partitions } \lambda \\ 0 < |\lambda| \leq n \end{array} \right\}$

__Pf.__ This follows by an obvious inductive argument from our previous discussion of idempotents, reducing to the base case of $\mathrm{Par}_1(t)$, which has idempotents $t^{-1}(\vdots)$, $1 - t^{-1}(\vdots)$ if $t \neq 0$ (here $\vdots$ is the diagram in $\mathrm{Par}_{1,1}$ consisting of two disconnected vertices. It satisfies $(\vdots)^2 = t(\vdots)$). If $t = 0$, we get $\mathrm{Par}_1(t) \cong \mathbb{C}[x]/(x^2)$ which has only one idempotent, i.e. 1.

This lets us describe some indecomposable objects of $\underline{\mathrm{Rep}}(S_t)$.

__Thm__ $\left\{ \begin{array}{l} \text{nonzero indec. objects of } \underline{\mathrm{Rep}}(S_t) \\ \text{of form } ([m], e) \text{ with } m \leq n \text{ up to iso.} \end{array} \right\} \overset{\text{bij.}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{partitions } \lambda \\ 0 \leq |\lambda| \leq n \end{array} \right\}$

We write L_λ for the indecomposable associated to λ.

__Pf.__ We have for $0 < |\lambda| \leq n$, some $e \in \mathrm{Par}_n(t)$ s.t. $([n], e) \cong L_\lambda$. If $t \neq 0$, also get $([n], e) \cong L_\varnothing$, but if $t = 0$ this only happens for $n = 0$.
Note that the diagrams ,  provide inverse isomorphisms between $\varphi \mathrm{Par}_n(t) \varphi$ and $\mathrm{Par}_{n-1}(t)$ (as before), giving $([n], e) \cong ([n-1], e')$ (for some e') in that case.
(i.e. e not coming from an idempotent of $\mathbb{C}S_n$)

So at each value of m we pick up indecomposables for $\lambda \vdash m$, ultimately giving λ for $0 \le |\lambda| \le n$.

Cor. $\left\{ \begin{array}{l} \text{nonzero indec. objects} \\ \text{of } \underline{\mathrm{Rep}}(S_t) \text{ up to iso.} \end{array} \right\} \overset{\text{bij.}}{\longleftrightarrow} \{ \text{partitions} \}$

Actually we have a "filtration" on these objects w.r.t. the tensor structure; L_λ has ~~a~~ the property that it is a summand of $[n]$ for $n = |\lambda|$ but no smaller n. It easily follows that $L_\lambda \otimes L_\mu$ should decompose as a sum of summands of $[|\lambda| + |\mu|]$, i.e. L_ν with $|\nu| \le |\lambda| + |\mu|$.

Now we'll discuss the tensor structure. Recall that $\mathrm{Par}_n(t)$ is semisimple for $t \notin \{0, 1, \dots, n-1\}$. This means that the subcategory of $\underline{\mathrm{Rep}}(S_t)$ additively generated (i.e. by direct sums) by objects L_λ where $|\lambda| < t \in \mathbb{N}$ is semisimple (though it doesn't inherit a tensor structure, as the subcategory won't be closed under $\otimes$).
Now recall that if $t \in \mathbb{N}$, we have a $\otimes$-functor $\mathcal{F} : \underline{\mathrm{Rep}}(S_t) \to \mathrm{Rep}(S_t)$ defined by $V \mapsto$ permutation rep^{n}. The functor is realised as "quotienting by negligible morphisms" and it is full and essentially surjective.

Thm (Deligne) If $t - |\lambda| \ge \lambda_1$ then $\mathcal{F}(L_\lambda) = S^{\lambda(t)}$ and otherwise $\mathcal{F}(L_\lambda) = 0$.

Here, $\lambda(t) = (t - |\lambda|, \lambda_1, \lambda_2, \dots)$ and $t - |\lambda| \ge \lambda_1$ is the condition for this to be a partition. S^μ is the Specht module associated to $\mu \vdash t$.

We won't prove this, but we'll sketch some aspects:

Since $\mathcal{F}$ is a full, essentially surjective $\otimes$-functor, the image of a simple object is either simple or zero, and distinct simple objects in $\underline{Rep}(S_t)$ have distinct (i.e. nonisomorphic) images if the images are nonzero. (For example if there is a nonzero map between $\mathcal{F}(L_\lambda)$ and $\mathcal{F}(L_\mu)$, by fullness there is a nonzero map between L_λ and L_μ; if also $|\lambda|, |\mu| < t$, L_λ and L_μ are simple, so they are isomorphic.)

We now work in $Rep(S_t)$, letting M be the permutation representation. We'll be interested in which irreducibles appear in $M^{\otimes k}$ but not in $M^{\otimes l}$ for $l<k$. If N is a rep^n of S_t, we'll need the following:

$$\underset{\substack{\uparrow \\ \text{character}}}{\mathrm{Char}}(M \otimes N)(g) = \underbrace{\mathrm{Char}(M)(g)}\cdot \mathrm{Char}(N)(g)$$

$g \in S_t$

character of permutation rep^n assigns to g the number of fixed points of g (acting on $\{1, 2, \ldots, t\}$).

Write T_μ for the indicator f^n of the conjugacy class of elements of cycle type μ ($\mu \vdash t$). It is a class function which we may restrict to S_{t-1} and then induce back to S_t. We do this using the Frobenius character formula, noting that we may take $(1\ t), (2\ t), \ldots, (t\text{-}1\ t), (t\ t) = \mathrm{id}$ to be left coset representatives for S_{t-1} in S_t.

$$\mathrm{Ind}_{S_{t-1}}^{S_t}\left(\mathrm{Res}_{S_{t-1}}^{S_t}(T_\mu)\right)(g) = \sum_{\substack{x \in S_t/S_{t-1} \\ xgx^{-1} \in S_{t-1}}} \mathrm{Res}_{S_{t-1}}^{S_t}(T_\mu)(g).$$

given our choice of coset rep$^{\underline{ns}}$ x, xgx^{-1} will be in S_{t-1} (let $x = (i\ t)$) iff i is a fixed point of g.
Since T_μ is zero off the conjugacy class of cycle type μ, we clearly obtain the following:

$$\mathrm{Ind}_{S_{t-1}}^{S_t}\left(\mathrm{Res}_{S_{t-1}}^{S_t}(T_\mu)\right)(g) = T_\mu(g) \cdot \{\#\text{ of fixed points of } g\}$$

We multiply by the value of the character of N on an element of cycle type μ, and sum over μ to see:

$$\begin{aligned}\mathrm{Ind}_{S_{t-1}}^{S_t}\left(\mathrm{Res}_{S_{t-1}}^{S_t}(\mathrm{Char}(N))\right)(g) &= \mathrm{Char}(N)(g) \cdot \{\#\text{ fixed pts of } g\} \\ &= \mathrm{Char}(N \otimes M)(g).\end{aligned}$$

So we know that $N \otimes M \cong \mathrm{Ind}_{S_{t-1}}^{S_t}\left(\mathrm{Res}_{S_{t-1}}^{S_t}(N)\right)$.

Now recall the branching rules for rep$^{\underline{ns}}$ of symmetric gps.

$$\mathrm{Res}_{S_{t-1}}^{S_t}(S^\lambda) = \bigoplus_{\mu \to \lambda} S^\mu, \quad \mathrm{Ind}_{S_{t-1}}^{S_t}(S^\mu) = \bigoplus_{\mu \to \lambda} S^\lambda$$

Here $\mu \to \lambda$ means μ is obtained from λ by removing a single box (or λ is obtained from μ by adding a single box).

eg $\mathrm{Res}_{S_4}^{S_5}(S^{(3,2)}) = S^{(2,2)} \oplus S^{(3,1)}$, $\mathrm{Ind}_{S_4}^{S_5}(S^{(3,1)}) = S^{(3,1,1)} \oplus S^{(3,2)} \oplus S^{(4,1)}$

So $\mathcal{F}(V^{\otimes 0}) = M^{\otimes 0}$ = trivial rep^{n} $= S^{(t)}$

also $\mathcal{F}(V^{\otimes 1}) = M^{\otimes 1} = S^{(t)} \oplus S^{(t-1,1)}$

we could find $\mathcal{F}(V^{\otimes 2}) = M^{\otimes 2} = (S^{(t)} \oplus S^{(t-1,1)}) \oplus (S^{(t)} \oplus S^{(t-1,1)} \oplus S^{(t-1,1)} \oplus S^{(t-2,2)} \oplus S^{(t-2,1,1)})$

Note that we need $t \geq 4$ for this to be right. (else $(t-2,2)$ won't be a partition.)

By the explicit description of Induction/Restriction, it's easy to see that each operation of "remove one box, and put it back somewhere" (describing the functor $M \otimes -$) can increase the number of boxes below the first row by at most 1. In particular since $M^{\otimes 0} = S^{(t)}$ has zero boxes below the first row, and an easy induction gives that the irreducibles appearing in $M^{\otimes k}$ that do not occur in $M^{\otimes l}$ for $l < k$ are all $S^{\lambda(t)}$ with $|\lambda| = k$, we compare summands of $\mathcal{F}(V^{\otimes k}) = \mathcal{F}(V)^{\otimes k} = M^{\otimes k}$ to see that $\{L_\lambda \mid |\lambda| = k\} \longleftrightarrow \{S^{\lambda(t)} \mid |\lambda| = k\}$ (each object on the right must come from a unique object on the left). We will not show that the indexing of the two sets is compatible with the bijection. (Note: all this assumes t is sufficiently large compared to k).

This fact has the following consequence for the tensor structure. Let us try to decompose $L_\mu \otimes L_\nu$, and apply $\mathcal{F}$ (where t is large relative to $|\mu| + |\nu|$).

$$\mathcal{F}(L_\mu \otimes L_\nu) = \mathcal{F}(L_\mu) \otimes \mathcal{F}(L_\nu).$$
$$= S^{\mu(t)} \otimes S^{\nu(t)} = \bigoplus_\lambda k_{\mu\nu}^{\lambda} S^{\lambda(t)}.$$
$$= \bigoplus_\lambda k_{\mu\nu}^{\lambda} \mathcal{F}(L_\lambda)$$

Here one would expect that $k_{\mu,\nu}^{\lambda}$ would have to be constant as a function of t for t sufficiently large and in fact this property holds ("stability of Kronecker coefficients"). When we identify $S^{\lambda(t)}$ with $\mathcal{F}(L_\lambda)$, we see that $k_{\mu\nu}^{\lambda}$ are the structure constants of the tensor product in $\underline{Rep}(S_t)$.

i.e. $L_\mu \otimes L_\nu \cong \bigoplus_\lambda (L_\lambda)^{\oplus k_{\mu\nu}^{\lambda}}$

Centre of $\underline{Rep}(S_t)$.

We'll be interested in the block structure of $\underline{Rep}(S_t)$ (we already know $\underline{Rep}(S_t)$ is semisimple unless $t = 0, 1, 2, \dots$)

Note: blocks in $\underline{Rep}(S_t)$ are (the finest possible) equivalence classes of indecomposable objects generated by $L_\mu \sim L_\lambda$ if there is a nonzero map $L_\mu \to L_\lambda$.

We're going to try to construct some endomorphisms of the identity functor which will allow us to test if two objects are in the same block.

eg. Suppose $E : L_\lambda \to L_\lambda$ is multiplication by a scalar E_λ (i.e. composing with a morphism (on either side) multiplies the morphism by E_λ)
Then if $E_\lambda \neq E_\mu$ there can be no nonzero maps between L_λ and L_μ and it's easy to see L_λ, L_μ must be in different blocks.

Recall $\mathrm{Par}_n(m) \cong \mathrm{End}_{S_m}((\mathbb{C}^m)^{\otimes n})$ for $m \gg n$.
We'll construct elements of $\mathrm{Par}_n(m)$ corresponding to central elements of $\mathrm{End}_{S_m}((\mathbb{C}^m)^{\otimes n})$. An obvious choice of central element is the action of an element of $Z(\mathbb{C}S_m)$, and this is spanned by sums of conjugacy classes (e.g. sum of all transpositions)

Note: $Z(\mathbb{C}S_m)$ can be given a filtration by saying that the k^{th} filtered component, F_k, is spanned by Ω_μ for $|\mu| - m_1(\mu) \leq k$, where Ω_μ is the sum of all elements of S_m of cycle type μ, and $m_1(\mu)$ is the number of parts of μ equal to 1. (so $|\mu| - m_1(\mu)$ measures the number of elements of $\{1, 2, \ldots, t\}$ that are not fixed by an element of cycle type μ; this makes it clear that $F_\ell \cdot F_k \subseteq F_{\ell+k}$, as moving ℓ objects, then k moves at most $\ell+k$ in total (but some may be moved more than once)). We abbreviate $\Omega_{(r, 1^{m-r})}$ to $\Omega_{(r)}$.

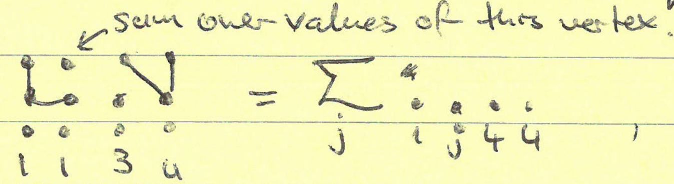
Now if $\mu = 1^{m_1} 2^{m_2} \cdots r^{m_r}$, then

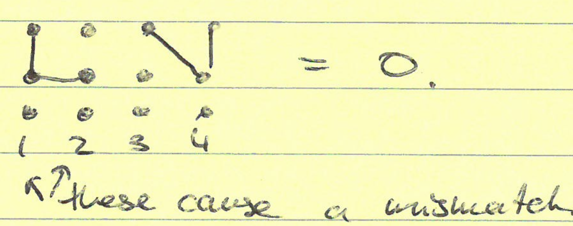
$$\Omega_\mu = \frac{\Omega_{(r)}^{m_r} \Omega_{(r-1)}^{m_{r-1}} \cdots \Omega_{(2)}^{m_2}}{(m_r!) \cdot (m_{r-1}!) \cdot \ldots \cdot (m_2!)} + \text{lower order terms}$$

Which shows that $\Omega_{(2)}, \Omega_{(3)}, \ldots, \Omega_{(m)}$ generate $\mathrm{Gr}_F(Z(\mathbb{C}S_m))$ and hence $Z(\mathbb{C}S_m)$. So if you believe that <u>Rep</u>(S_t) is somehow like $\mathrm{Rep}(S_t)$, the centre will be adequately understood by constructing analogues of $\Omega_{(r)}$

Fun fact! it is possible to choose ~~a basis~~ elements (replacing $\{\Omega(m)\}$) of $Z(\mathbb{Z}S_m)$ that (integrally) generates $Z(\mathbb{Z}S_m)$ (in particular, make the denominators in the previous equation go away), so it's possible to do modular reduction in a specific sense (see "Farahat-Higman algebras") but it is more complicated and won't make our task any easier.

Recall that $Par_n(m)$ acts on $(\mathbb{C}^m)^{\otimes n}$ as follows. Let $e_1, e_2, \ldots, e_m$ be the std basis of $\mathbb{C}^m$, so $(\mathbb{C}^m)^{\otimes n}$ has a basis of form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$ where $i_j \in \{1, 2, \ldots, m\}$ we abbreviate this vector to $\vec{i}$, and draw it as a sequence of labelled vertices $\underset{i_1}{\cdot}\ \underset{i_2}{\cdot}\ \underset{i_3}{\cdot} \cdots \underset{i_n}{\cdot}$.
To apply the partition diagram π, we draw it above the vertices of $\vec{i}$ and label each part with the label of any vertex in the diagram of $\vec{i}$ that it touches. If this labelling is inconsistent, we get zero, otherwise we get the top row with labelling coming from the labellings of the parts (unlabelled parts are summed over all possible values) eg.

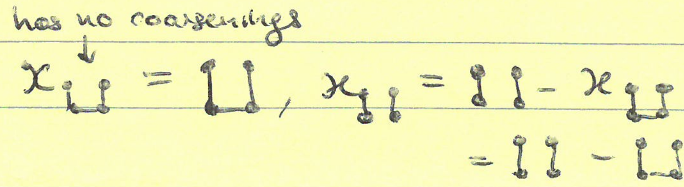




Now we inductively define a basis of $Par_n(m)$ via

$$x_\pi = \pi - \sum_{\mu \text{ coarsening of } \pi} x_\mu$$

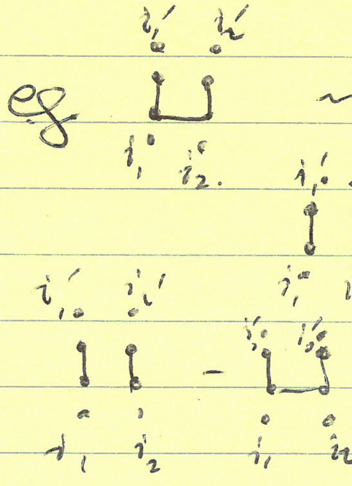
eg.



The point of this construction is that, firstly, as coarsening of partitions is a partial order, the x_π are related to the π by a unitriangular matrix, so they indeed form a basis of $\mathrm{Par}_n(m)$. Secondly the multiplicity of $\vec{i'}$ in $x_\pi \vec{i}$ admits the following easy description:

take the diagram of π, and label the parts with the numbering of the vertices of $\vec{i}$ below, and the numbering of the vertices of $\vec{i'}$ above. If this diagram is consistently labelled <u>and</u> distinct parts have different labels, then the multiplicity is 1, otherwise it is zero.

This is shown by induction on the poset of partitions (ordered by coarsening).



e.g. $\rightsquigarrow$ 1 if $i_1' = i_2' = i_1 = i_2$.

$\rightsquigarrow$ 1 if $i_1 = i_1'$, $i_2 = i_2'$

$\rightsquigarrow$ 1 if $i_1 = i_1' \neq i_2 = i_2'$

corresp. to two parts of ⁞ ⁞

Now we'll use this to construct elements of $\mathrm{Par}_n(t)$ such that when $t = m \in \mathbb{N}$, the image of the element under F (hence, an element of $\mathrm{End}_{S_m}((\mathbb{C}^m)^{\otimes n})$) is precisely $\Omega_{(\pi)}$.

We consider the following.

$w_n^r(t) = \sum_{\text{partitions } \pi} \{\#$ r-cycles $\sigma \in S_t$ s.t. if distinct parts of π are given distinct labels in $\{1,2,\dots,t\}$, in a fixed but arbitrary way, then if the top row of vertices has label $\vec{i}'$ and the bottom has label $\vec{i}$ then $\sigma(\vec{i}) = \sigma(\vec{i}')$ where the action is componentwise i.e. $\sigma(i_j) = i'_j \} \cdot x_\pi$

Note that the horrible mess that is the coefficient of x_π is precisely the right quantity so that $F(w_n^r(t)) = \Omega_{(r)}$, i.e. it describes the action of the sum of all r-cycles acting on $(\mathbb{C}^t)^{\otimes n}$.

Fact: some rather annoying (but unsophisticated) combinatorics shows that these coefficients are actually polynomials in t.

This means we can make sense of this definition when $t \notin \mathbb{N}$, just by substituting the relevant polynomial evaluated at t.

——— End of material covered in lecture ———

(By playing with the functor F, one can show $w_n^r(t)$ are central (define endomorphisms of the identity functor) and calculate the action on L_λ via character formulae for symmetric groups; this will allow us to describe the blocks).